Appendix to

"The Econometrics of the Hodrick-Prescott filter"

by

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and

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Appendix 2: Mathematical proofs

In this section, C and C_1, C_2,... denote arbitrary constants that can take different values in different proofs.

Lemma 2. *Let* $p_{Tt} = (p_1(t/T), \ldots, p_T(t/T))'$, *where* $p_1(t/T) = 1$ *and* $p_j(t/T) = \sqrt{2}\cos(\pi(j-1)(t-1/2)/T)$, $j = 2, \ldots, T$. *Then* $T^{-1}\sum_{t=1}^{T} p_{Tt}p'_{Tt} = I_T$.

Proof. First note that for $k \in \{1, 2, \ldots, 2T-1\}$,

$$A(k,T) = \sum_{t=1}^{T} \cos(\pi k(t-1/2)/T) = 0.$$

To see this, note that for $k \in \{1, 2, \ldots, 2T-1\}$, because $\sum_{t=1}^{T} \rho^{t-1/2} = (\rho-1)^{-1}\rho^{1/2}(\rho^T - 1)$ for $\rho \neq 1$,

$$\sum_{t=1}^{T} \exp(i\pi k(t-1/2)/T) = (\exp(i\pi k/T) - 1)^{-1}\exp((1/2)i\pi k/T)(\exp(i\pi k) - 1)$$

and

$$\sum_{t=1}^{T} \exp(-i\pi k(t-1/2)/T) = (\exp(-i\pi k/T) - 1)^{-1}\exp(-(1/2)i\pi k/T)(\exp(-i\pi k) - 1).$$

Now for k even, $\exp(-i\pi k) = \exp(i\pi k) = 1$ while for k odd, $\exp(-i\pi k) = \exp(i\pi k) = -1$. Therefore, for $k \in \{1, 2, \ldots, 2T-1\}$ even, by Euler's formula,

$$A(k,T) = (1/2)\sum_{t=1}^{T} \exp(i\pi k(t-1/2)/T) + (1/2)\sum_{t=1}^{T} \exp(-i\pi k(t-1/2)/T)$$

$$= 0 + 0 = 0,$$

while for $k \in \{1, 2, \ldots, 2T-1\}$ odd,

$$A(k,T) = (1/2)\sum_{t=1}^{T} \exp(i\pi k(t-1/2)/T) + (1/2)\sum_{t=1}^{T} \exp(-i\pi k(t-1/2)/T)$$

$$= (1/2)(\exp(i\pi k/T) - 1)^{-1}\exp((1/2)i\pi k/T)(-2) + (1/2)(\exp(-i\pi k/T) - 1)^{-1}\exp(-(1/2)i\pi k/T)(-2),$$

and setting $z = \exp((1/2)i\pi k/T)$, it now follows that the last expression equals 0 because

$$-(z^2-1)^{-1}z - (z^{-2}-1)^{-1}z^{-1} = 0.$$

Therefore, $A(k,T) = 0$ for $k \in \{1, 2, \ldots, 2T-1\}$. To prove the lemma, first note that trivially, $T^{-1}\sum_{t=1}^{T} p_1(t/T)p_1(t/T) = T^{-1}\sum_{t=1}^{T} 1 = 1$. For $j \in \{2, 3, \ldots, T\}$, by the earlier property of $A(k,T)$,

$$T^{-1}\sum_{t=1}^{T} p_j(t/T)p_1(t/T) = T^{-1}\sum_{t=1}^{T} \sqrt{2}\cos(\pi(j-1)(t-1/2)/T)$$

$$= \sqrt{2}T^{-1}A(j-1,T) = 0,$$

and therefore, for $j \in \{2, 3, \ldots, T\}$,

$$\left[T^{-1}\sum_{t=1}^{T} p_{Tt}p_{Tt}'\right]_{j,1} = \left[T^{-1}\sum_{t=1}^{T} p_{Tt}p_{Tt}'\right]_{1,j} = 0.$$

Furthermore, by the trigonometric identity $\cos(\alpha)\cos(\beta) = (\cos(\alpha+\beta) + \cos(\alpha-\beta))/2$, for

$j, k \in \{2, 3, \ldots, T\}$,

$$\left[T^{-1}\sum_{t=1}^{T} p_{Tt}p'_{Tt}\right]_{j,k} = T^{-1}\sum_{t=1}^{T} p_j(t/T)p_k(t/T)$$

$$= T^{-1}\sum_{t=1}^{T}\cos(\pi(j+k-2)(t-1/2)/T) + T^{-1}\sum_{t=1}^{T}\cos(\pi(j-k)(t-1/2)/T)$$
$$= T^{-1}A(j+k-2, T) + T^{-1}A(|j-k|, T).$$

The above expression is zero for $j \neq k$ by the earlier result on $A(k, T)$, while for $j = k$, $T^{-1}A(0, T) = 1$, which shows the result. □

Lemma 3. *Assume that* $h : [0, 1] \to \mathbb{R}$ *is continuous on* $[0, 1]$*. Then*

$$\lim_{T\to\infty} T^{-1}\sum_{j=1}^{T} h((j-1)/T) = \int_0^1 h(r)dr.$$

Proof. This follows because by continuity of $h(\cdot)$ and setting $j = rT$,

$$T^{-1}\sum_{j=1}^{T} h((j-1)/T) = T^{-1}\int_0^T h([j]/T)dj = \int_0^1 h([rT]/T)dr,$$

and because $\sup_{r\in[0,1]} |h(r)| < \infty$ by continuity and because $r - 1/T \leq [rT]/T \leq r$, the result now follows by the dominated convergence theorem. □

Lemma 4. *Assume that* $h : [0,1] \to \mathbb{R}$ *is continuous on* $[0,1]$*. Then*

$$T^{-1}\sum_{j=2}^{T}\cos(\pi(j-1)m/T)h((j-1)/T)$$

$$= -(2T)^{-1}h(1/T)$$

$$-(2T)^{-1}(-1)^m h((T-1)/T)$$

$$-T^{-1}(2\sin(\pi m/(2T))^{-2}(h(2/T)-h(1/T))\cos(\pi m/T)$$

$$+T^{-1}(2\sin(\pi m/(2T))^{-2}(h(1-1/T)-h(1-2/T))\cos(\pi m(1-1/T))$$

$$+T^{-1}(2\sin(\pi m/(2T)))^{-3}(h(3/T)-2h(2/T)+h(1/T))\sin(\pi(3/2)m/T)$$

$$-T^{-1}(2\sin(\pi m/(2T)))^{-3}(h(1-1/T)-2h(1-2/T)+h(1-3/T))\sin(\pi(T-3/2)m/T)$$

$$+T^{-1}(2\sin(\pi m/(2T)))^{-3}\sum_{j=2}^{T-3}\Delta^3 h((j+2)/T)\sin(\pi(j+1/2)m/T)$$

and

$$T^{-1}\sum_{j=1}^{T}\cos(\pi(j-1)(m-(1/2))/T)\cos(\pi(j-1)/(2T))h((j-1)/T)$$

$$= (T)^{-1}h(0) - (2T)^{-1}h(1/T)$$

$$-(2T)^{-1}(h(2/T)-h(1/T))\left\{\frac{\cos(\pi m/T)}{(2\sin(\pi m/(2T)))^2}+\frac{\cos(\pi(m-1)/T)}{(2\sin(\pi(m-1)/(2T)))^2}\right\}$$

$$+(2T)^{-1}(h(1-1/T)-h(1-2/T))\left\{\frac{\cos(\pi m(1-1/T))}{(2\sin(\pi m/(2T)))^2}+\frac{\cos(\pi(m-1)(1-1/T))}{(2\sin(\pi(m-1)/(2T)))^2}\right\}$$

$$+(2T)^{-1}\Delta^2 h(3/T)\left\{\frac{\sin(3\pi m/(2T))}{(2\sin(\pi m/(2T)))^3}+\frac{\sin(3\pi(m-1)/(2T))}{(2\sin(\pi(m-1)/(2T)))^3}\right\}$$

$$-(2T)^{-1}\Delta^2 h(1-1/T)\left\{\frac{\sin(\pi m(T-3/2)/T)}{(2\sin(\pi m/(2T)))^3}+\frac{\sin(\pi(m-1)(T-3/2)/T)}{(2\sin(\pi(m-1)/(2T)))^3}\right\}$$

$$+(2T)^{-1}\sum_{j=2}^{T-3}\Delta^3 h((j+2)/T)\left\{\frac{\sin(\pi(j+(1/2))m/T)}{(2\sin(\pi m/(2T)))^3}+\frac{\sin(\pi(j+(1/2))(m-1)/T)}{(2\sin(\pi(m-1)/(2T)))^3}\right\}.$$

Proof. The proof of this result is somewhat tedious and can be found at `https://dl.dropbox.com/u/2159931/hp.html`. □

Proof of Lemma 1: By Lemma 2, $T^{-1}\sum_{t=1}^{T} p_{Tt}p_{Tt}' = I_T$, and therefore it only remains to prove the second part of Lemma 1. To show this, note that for $j \in \{1, 2, ..., T\}$, remembering that $p_1(t/T) = 1$ for $t \in \mathbb{Z}$,

$$p_j(t/T) - p_j((t-1)/T) = \sqrt{2}\cos(\pi(j-1)(t-1/2)/T) - \sqrt{2}\cos(\pi(j-1)(t-3/2)/T),$$

and because

$$\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin(\alpha)\sin(\beta)$$

and setting $\alpha + \beta = \pi(j-1)(t-1/2)/T$ and $\alpha - \beta = \pi(j-1)(t-3/2)/T$, we get

$$p_j(t/T) - p_j((t-1)/T) = -2\sqrt{2}\sin(\pi(j-1)(t-1)/T))\sin(\pi(j-1)/(2T)).$$

Therefore,

$$p_j((t+1)/T) - 2p_j(t/T) + p_j((t-1)/T)$$

$$= (p_j((t+1)/T) - p_j(t/T)) - (p_j(t/T) - p_j((t-1)/T))$$

$$= 2\sqrt{2}\sin(\pi(j-1)/(2T))(\sin(\pi(j-1)(t-1)/T) - \sin(\pi(j-1)t/T)).$$

Now because

$$\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\sin(\beta)\cos(\alpha)$$

and setting $\alpha + \beta = \pi(j-1)(t-1)/T$ and $\alpha - \beta = \pi(j-1)t/T)$, we have

$$\sin(\pi(j-1)(t-1)/T) - \sin(\pi(j-1)t/T)$$

$$= -2\sin(\pi(j-1)/(2T))\cos(\pi(j-1)(t-1/2)/T).$$

Therefore, for $j \in \{1, 2, \ldots, T\}$ and $t \in \mathbb{Z}$,

$$p_j((t+1)/T) - 2p_j(t/T) + p_j((t-1)/T) = -4\sqrt{2}\sin(\pi(j-1)/(2T))^2\cos(\pi(j-1)(t-1/2)/T),$$

implying that, for $j, k \in \{1, 2, ..., T\}$,

$$\Big[\sum_{t=2}^{T-1}(p_{T,t+1} - 2p_{Tt} + p_{T,t-1})(p_{T,t+1} - 2p_{Tt} + p_{T,t-1})'\Big]_{jk}$$

$$= 32\sin(\pi(j-1)/(2T))^2 \sin(\pi(k-1)/(2T))^2 \sum_{t=2}^{T-1} \cos(\pi(j-1)(t-1/2)/T) \cos(\pi(k-1)(t-1/2)/T).$$

Also, for $j, k \in \{1, 2, \ldots, T\}$, by Lemma 2,

$$\sum_{t=2}^{T-1} \cos(\pi(j-1)(t-1/2)/T) \cos(\pi(k-1)(t-1/2)/T)$$

$$= (1/2)TI(j = k) - \cos(\pi(j-1)/(2T)) \cos(\pi(k-1)/(2T))$$

$$- \cos(\pi(j-1)(T-1/2)/T) \cos(\pi(k-1)(T-1/2)/T),$$

implying that

$$T^{-1}\sum_{t=2}^{T-1}(p_{T,t+1} - 2p_{Tt} + p_{T,t-1})(p_{T,t+1} - 2p_{Tt} + p_{T,t-1})'$$

$$= \mathrm{diag}(\{16\sin(\pi(j-1)/(2T))^4, j = 1, \ldots, T\}) - 32T^{-1}q_{T1}q_{T1}' - 32T^{-1}q_{T2}q_{T2}',$$

where $q_{T1j} = \sin(\pi(j-1)/(2T))^2 \cos(\pi(j-1)/(2T))$ and $q_{T2j} = \sin(\pi(j-1)/(2T))^2 \cos(\pi(j-1)(T-1/2)/T)$. □

Proof of Theorem 1: It follows from Lemma 1 that

$$\hat{\theta} = \Big[T^{-1}\sum_{t=1}^{T} p_{Tt}p'_{Tt} + \lambda T^{-1}\sum_{t=2}^{T-1}(p_{T,t+1}-2p_{Tt}+p_{T,t-1})(p_{T,t+1}-2p_{Tt}+p_{T,t-1})'\Big]^{-1} T^{-1}\sum_{t=1}^{T} y_t p_{Tt}$$

$$= \Big[I_T + \lambda D_T + H_T(I_T + \lambda D_T)\Big]^{-1} \ T^{-1}\sum_{t=1}^{T} y_t p_{Tt}$$

$$= (I_T + \lambda D_T)^{-1}(I_T + H_T)^{-1}T^{-1}\sum_{t=1}^{T} y_t p_{Tt}$$

where

$$H_T = (-32\lambda T^{-1}q_{T1}q'_{T1} - 32\lambda T^{-1}q_{T2}q'_{T2})(I_T + \lambda D_T)^{-1}.$$

Now according to Miller (1981), for a matrix H_T of rank 0, 1 or 2, we have

$$(I_T + H_T)^{-1} = I_T - (a_T + b_T)^{-1}(a_T H_T - H_T^2)$$

for $a_T = 1 + \mathrm{tr}(H_T)$ and $2b_T = (\mathrm{tr}(H_T))^2 - \mathrm{tr}(H_T^2)$. Therefore,

$$(I_T + \lambda D_T)^{-1}(I_T + H_T)^{-1} = (I_T + \lambda D_T)^{-1}(I_T - (a_T + b_T)^{-1}(a_T H_T - H_T^2))$$

$$= (I_T + \lambda D_T)^{-1}$$

$$+32\lambda a_T(a_T + b_T)^{-1}(I_T + \lambda D_T)^{-1}(T^{-1}q_{T1}q'_{T1})(I_T + \lambda D_T)^{-1}$$

$$+32\lambda a_T(a_T + b_T)^{-1}(I_T + \lambda D_T)^{-1}(T^{-1}q_{T2}q'_{T2})(I_T + \lambda D_T)^{-1}$$

$$+32^2\lambda^2(a_T+b_T)^{-1}(I_T+\lambda D_T)^{-1}(T^{-1}q_{T1}q'_{T1})(I_T+\lambda D_T)^{-1}(T^{-1}q_{T1}q'_{T1})(I_T+\lambda D_T)^{-1}$$

$$+32^2\lambda^2(a_T+b_T)^{-1}(I_T+\lambda D_T)^{-1}(T^{-1}q_{T1}q'_{T1})(I_T+\lambda D_T)^{-1}(T^{-1}q_{T2}q'_{T2})(I_T+\lambda D_T)^{-1}$$

$$+32^2\lambda^2(a_T+b_T)^{-1}(I_T+\lambda D_T)^{-1}(T^{-1}q_{T2}q'_{T2})(I_T+\lambda D_T)^{-1}(T^{-1}q_{T1}q'_{T1})(I_T+\lambda D_T)^{-1}$$

$$+32^2\lambda^2(a_T+b_T)^{-1}(I_T+\lambda D_T)^{-1}(T^{-1}q_{T2}q'_{T2})(I_T+\lambda D_T)^{-1}(T^{-1}q_{T2}q'_{T2})(I_T+\lambda D_T)^{-1}$$

$$=\sum_{i=1}^{7} M_{Ti},$$

say, and we can write

$$\hat{\tau}_{Tt}=p'_{Tt}\hat{\theta}=p'_{Tt}\sum_{i=1}^{7}M_{Ti}T^{-1}\sum_{s=1}^{T}p_{Ts}y_s=\sum_{s=1}^{T}y_s\sum_{i=1}^{7}T^{-1}p'_{Tt}M_{Ti}p_{Ts}$$

where

$$T^{-1}p'_{Tt}M_{T1}p_{Ts}=T^{-1}p'_{Tt}(I_T+\lambda D_T)^{-1}p_{Ts},$$

$$T^{-1}p'_{Tt}M_{T2}p_{Ts}=32\lambda a_T(a_T+b_T)^{-1}(T^{-1}p'_{Tt}(I_T+\lambda D_T)^{-1}q_{T1})(T^{-1}q'_{T1}(I_T+\lambda D_T)^{-1}p_{Ts}),$$

$$T^{-1}p'_{Tt}M_{T3}p_{Ts}=32\lambda a_T(a_T+b_T)^{-1}(T^{-1}p'_{Tt}(I_T+\lambda D_T)^{-1}q_{T2})(T^{-1}q'_{T2}(I_T+\lambda D_T)^{-1}p_{Ts}),$$

$$T^{-1}p'_{Tt}M_{T4}p_{Ts}$$

$$=32^2\lambda^2(a_T+b_T)^{-1}(T^{-1}p_{Tt}(I_T+\lambda D_T)^{-1}q_{T1})(T^{-1}q'_{T1}(I_T+\lambda D_T)^{-1}q_{T1})(T^{-1}q'_{T1}(I_T+\lambda D_T)^{-1}p_{Ts}),$$

$$T^{-1}p'_{Tt}M_{T5}p_{Ts}$$

$$=32^2\lambda^2(a_T+b_T)^{-1}(T^{-1}p_{Tt}(I_T+\lambda D_T)^{-1}q_{T1})(T^{-1}q'_{T1}(I_T+\lambda D_T)^{-1}q_{T2})(T^{-1}q'_{T2}(I_T+\lambda D_T)^{-1}p_{Ts}),$$

$$T^{-1}p'_{Tt}M_{T6}p_{Ts}$$

$$= 32^2\lambda^2(a_T{+}b_T)^{-1}(T^{-1}p_{Tt}(I_T{+}\lambda D_T)^{-1}q_{T2})(T^{-1}q'_{T2}(I_T{+}\lambda D_T)^{-1}q_{T1})(T^{-1}q'_{T1}(I_T{+}\lambda D_T)^{-1}p_{Ts}),$$

and

$$T^{-1}p'_{Tt}M_{T7}p_{Ts}$$

$$= 32^2\lambda^2(a_T{+}b_T)^{-1}(T^{-1}p_{Tt}(I_T{+}\lambda D_T)^{-1}q_{T2})(T^{-1}q'_{T2}(I_T{+}\lambda D_T)^{-1}q_{T2})(T^{-1}q'_{T2}(I_T{+}\lambda D_T)^{-1}p_{Ts}).$$

Now consider the first term $T^{-1}p'_{Tt}(I_T + \lambda D_T)^{-1}p_{Ts}$. Using the identity

$$\cos(\alpha)\cos(\beta) = (1/2)\cos(\alpha + \beta) + (1/2)\cos(\alpha - \beta),$$

one can write, for $j \geq 2$,

$$p_j(t/T)p_j(s/T) = 2\cos(\pi(j-1)(t-1/2)/T)\cos(\pi(j-1)(s-1/2)/T)$$

$$= \cos(\pi(j-1)(t+s-1)/T) + \cos(\pi(j-1)(t-s)/T).$$

Using the above equation, the first term can be represented as (remembering that $p_1(t/T) = 1$ for $t \in \{1, 2, \ldots, T\}$),

$$T^{-1}p'_{Tt}(I_T + \lambda D_T)^{-1}p_{Ts}$$

$$= T^{-1} + T^{-1}\sum_{j=2}^{T}\cos(\pi(j-1)(t+s-1)/T)(1 + 16\lambda\sin(\pi(j-1)/(2T))^4)^{-1}$$

$$+T^{-1}\sum_{j=2}^{T}\cos(\pi(j-1)(t-s)/T)(1+16\lambda\sin(\pi(j-1)/(2T))^4)^{-1}$$

$$=T^{-1}+f_{T\lambda}(t+s-1)-(2T)^{-1}-(2T)^{-1}(-1)^{t+s-1}(1+16\lambda)^{-1}$$

$$+f_{T\lambda}(t-s)-(2T)^{-1}-(2T)^{-1}(-1)^{t-s}(1+16\lambda)^{-1}$$

$$=f_{T\lambda}(t-s)+f_{T\lambda}(t+s-1)$$

because $(-1)^{t+s-1}+(-1)^{t-s}=(-1)^t((-1)^{s-1}+(-1)^s)=0$. By the result of Equation (19),

$$T^{-1}p'_{Tt}M_{T1}p_{Ts}=f_{T\lambda}(t-s)+f_{T\lambda}(T)I(t+s-1=T)$$

$$+f_{T\lambda}(s+t-1)I(t+s-1<T)+f_{T\lambda}(2T-t-s+1)I(t+s-1>T)=w^1_{Tts}+w^2_{Tts}+w^3_{Tts}+w^4_{Tts}.$$

Next, we consider the other terms and we will show that

$$\sum_{i=2}^{7}T^{-1}p'_{Tt}M_{Ti}p_{Ts}=w^5_{Tts}+w^6_{Tts}+w^7_{Tts}+w^8_{Tts}.$$

First note that

$$T^{-1}p'_{Tt}M_{T1}q_{T1}=g_{T\lambda}(t)$$

and by symmetry of D_T and I_T, $T^{-1}q'_{T1}(I_T+\lambda D_T)^{-1}q_{T2}=T^{-1}q'_{T2}(I_T+\lambda D_T)^{-1}q_{T1}=\eta_{T\lambda}$. Additionally,

$$T^{-1}p'_{Tt}M_{T1}q_{T2}=g_{T\lambda}(T-t+1).$$

To see this, consider that for $j \in \mathbb{Z}$

$$q_{T2j} = \cos(\pi(j-1))q_{T1j}.$$

Because $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$ and $\sin(\pi(j-1)) = 0$ for $j \in \mathbb{Z}$, it follows that

$$q_{T2j} = \sin(\pi(j-1)/(2T))^2 \cos(\pi(j-1) - \pi(j-1)/(2T))$$

$$= \sin(\pi(j-1)/(2T))^2 \cos(\pi(j-1))\cos(\pi(j-1)/(2T)) = \cos(\pi(j-1))q_{T1j}.$$

In addition, note that

$$p_j((T-t+1)/T) = \cos(\pi(j-1))p_j(t/T)$$

because

$$p_j((T-t+1)/T) = \sqrt{2}\cos(\pi(j-1)((T-t+1/2)/T))$$

$$= \sqrt{2}\cos(\pi(j-1))\cos(\pi(j-1)(t-1/2)/T) = \cos(\pi(j-1))p_j(t/T),$$

and therefore it follows that, defining $d_{jj} = D_{T,jj}$,

$$T^{-1}p'_{Tt}(I_T + \lambda D_T)^{-1}q_{T2} = T^{-1}\sum_{j=1}^{T} p_j(t/T)q_{T2j}(1+\lambda d_{jj})^{-1}$$

$$= T^{-1} \sum_{j=1}^{T} p_j(t/T) \cos(\pi(j-1)) q_{T1j} (1 + \lambda d_{jj})^{-1}$$

$$= T^{-1} p'_{T,T-t+1} (I_T + \lambda D_T)^{-1} q_{T1} = g_{T\lambda}(T - t + 1).$$

Finally, note that

$$T^{-1} q'_{T2} (I_T + \lambda D_T)^{-1} q_{T2} = T^{-1} q'_{T1} (I_T + \lambda D_T)^{-1} q_{T1} = \delta_T.$$

Therefore,

$$T^{-1} p'_{Tt} M_{T2} p_{Ts} = 32 \lambda a_T (a_T + b_T)^{-1} g_{T\lambda}(t) g_{T\lambda}(s),$$

$$T^{-1} p'_{Tt} M_{T3} p_{Ts} = 32 \lambda a_T (a_T + b_T)^{-1} g_{T\lambda}(T - t + 1) g_{T\lambda}(T - s + 1),$$

$$T^{-1} p'_{Tt} M_{T4} p_{Ts} = 32^2 \lambda^2 (a_T + b_T)^{-1} \delta_T g_{T\lambda}(t) g_{T\lambda}(s),$$

$$T^{-1} p'_{Tt} M_{T5} p_{Ts} = 32^2 \lambda^2 (a_T + b_T)^{-1} \eta_T g_{T\lambda}(t) g_{T\lambda}(T - s + 1),$$

$$T^{-1} p'_{Tt} M_{T6} p_{Ts} = 32^2 \lambda^2 (a_T + b_T)^{-1} \eta_T g_{T\lambda}(T - t + 1) g_{T\lambda}(s),$$

$$T^{-1} p'_{Tt} M_{T7} p_{Ts} = 32^2 \lambda^2 (a_T + b_T)^{-1} \delta_T g_{T\lambda}(T - t + 1) g_{T\lambda}(T - s + 1).$$

Now note that

$$a_T = 1 + \mathrm{tr}(H_T) = 1 + \mathrm{tr}((-32 \lambda T^{-1} q_{T1} q'_{T1} - 32 \lambda T^{-1} q_{T2} q'_{T2})(I_T + \lambda D_T)^{-1})$$

$$= 1 - 32 \lambda T^{-1} q'_{T1} (I_T + \lambda D_T)^{-1} q_{T1} - 32 \lambda T^{-1} q'_{T2} (I_T + \lambda D_T)^{-1} q_{T2}$$

$$= 1 - 64\lambda\delta_T,$$

$$b_T = (1/2)(\mathrm{tr}(H_T))^2 - (1/2)\mathrm{tr}(H_T^2)$$

$$= (1/2)(-64\lambda\delta_T)^2 - (1/2)32^2\lambda^2(2\delta_T^2 + 2\eta_T^2)$$

$$= 32^2\lambda^2(\delta_T^2 - \eta_T^2),$$

$$\xi_{T\lambda} = 32\lambda a_T(a_T + b_T)^{-1} + 32^2\lambda^2(a_T + b_T)^{-1}\delta_T,$$

and

$$\phi_{T\lambda} = 32^2\lambda^2(a_T + b_T)^{-1}\eta_T.$$

Therefore, setting

$$w_{Tts}^5 = \xi_{T\lambda}g_{T\lambda}(t)g_{T\lambda}(s)$$

$$w_{Tts}^6 = \phi_{T\lambda}g_{T\lambda}(T - t + 1)g_{T\lambda}(s)$$

$$w_{Tts}^7 = \phi_{T\lambda}g_{T\lambda}(t)g_{T\lambda}(T - s + 1)$$

$$w_{Tts}^8 = \xi_{T\lambda}g_{T\lambda}(T - t + 1)g_{T\lambda}(T - s + 1)$$

it now follows that

$$\sum_{i=2}^{7} T^{-1}p_{Tt}'M_{Ti}p_{Ts} = w_{Tts}^5 + w_{Tts}^6 + w_{Tts}^7 + w_{Tts}^8,$$

which completes the proof of the representation for w_{Tts}.

In order to find the upper bounds for $f_{T\lambda}(m)$ and $g_{T\lambda}(m)$, note that by the first result of Lemma 4, it follows that for $h(r) = 1/(1 + 16\lambda \sin(\pi r/2)^4)$,

$$f_{T\lambda}(m) = -(2T)^{-1}(h(1/T) - h(0))$$

$$-(2T)^{-1}(-1)^m(h((T-1)/T) - h(1))$$

$$-T^{-1}(2\sin(\pi m/(2T))^{-2}(h(2/T) - h(1/T))\cos(\pi m/T)$$

$$+T^{-1}(2\sin(\pi m/(2T))^{-2}(h(1-1/T) - h(1-2/T))\cos(\pi m(1-1/T))$$

$$+T^{-1}(2\sin(\pi m/(2T)))^{-3}(h(3/T) - 2h(2/T) + h(1/T))\sin(\pi(3/2)m/T)$$

$$-T^{-1}(2\sin(\pi m/(2T)))^{-3}(h(1-1/T) - 2h(1-2/T) + h(1-3/T))\sin(\pi(T-3/2)m/T)$$

$$+T^{-1}(2\sin(\pi m/(2T)))^{-3}\sum_{j=2}^{T-3}\Delta^3 h((j+2)/T)\sin(\pi(j+1/2)m/T).$$

Noting that $h'(0) = h'(1) = 0$ and that $\sup_{x\in[0,1]} |h''(x)| < \infty$, it follows from a Taylor series expansion of order 2 that the first and second terms are bounded in absolute value by

$$(1/4)T^{-3} \sup_{x\in[0,1]} |h''(x)|.$$

Similarly, the third and fourth terms are bounded by a multiple of

$$T^{-3}(2\sin(\pi m/(2T))^{-2},$$

and the fifth, sixth and seventh terms by a multiple of

$$T^{-3}(2\sin(\pi m/(2T))^{-3}.$$

By noting that

$$\sup_{r\in[0,1]} |r/\sin(\pi r/2)| = 1$$

and that $m \leq T$, it now follows that $|f_{T\lambda}(m)| \leq Cm^{-3}$. By using the second result of Lemma 4 and setting $h(r) = \sin(\pi r/2)^2(1 + 16\lambda\sin(\pi r/2)^4)^{-1}$, the bound for $g_{T\lambda}(m)$ can be proven analogously. □

Proof of Theorem 2: By Lemma 3,

$$\delta_{T\lambda} = T^{-1}q'_{T1}(I_T + \lambda D_T)^{-1}q_{T1}$$

$$= T^{-1}\sum_{j=1}^{T}\sin(\pi(j-1)/(2T))^4\cos(\pi(j-1)/(2T))^2(1 + 16\lambda\sin(\pi(j-1)/(2T))^4)^{-1}$$

$$\to \int_0^1 \sin(\pi r/2)^4\cos(\pi r/2)^2(1 + 16\lambda\sin(\pi r/2)^4)^{-1}dr = \delta_\lambda$$

and

$$\eta_{T\lambda} = T^{-1}q'_{T1}(I_T + \lambda D_T)^{-1}q_{T2}$$

$$= T^{-1} \sum_{j=1}^{T} q_{T1j} q_{T2j} (1 + 16\lambda \sin(\pi(j-1)/(2T))^4)^{-1},$$

and because $q_{T2j} = \cos(\pi(j-1)) q_{T1j}$,

$$\eta_{T\lambda} = T^{-1} \sum_{j=1}^{T} \cos(\pi(j-1)) q_{T1j}^2 (1 + 16\lambda \sin(\pi(j-1)/(2T))^4)^{-1}$$

$$= T^{-1} \sum_{j=1, j \text{ odd}}^{T} q_{T1j}^2 (1 + 16\lambda \sin(\pi(j-1)/(2T))^4)^{-1}$$

$$-T^{-1} \sum_{j=1, j \text{ even}}^{T} q_{T1j}^2 (1 + 16\lambda \sin(\pi(j-1)/(2T))^4)^{-1}$$

$$= T^{-1} \sum_{j=0}^{[(T-1)/2]} \sin(\pi j/T)^4 \cos(\pi j/T)^2 (1 + 16\lambda \sin(\pi j/T)^4)^{-1}$$

$$-T^{-1} \sum_{j=1}^{[T/2]} \sin(\pi(2j-1)/(2T))^4 \cos(\pi(2j-1)/(2T))^2 (1 + 16\lambda \sin(\pi(2j-1)/(2T))^4)^{-1},$$

and by an argument similar to that of Lemma 3, it now follows that both terms converge to the same number, implying that $\lim_{T\to\infty} \eta_{T\lambda} = 0$. Note that since $0 \le \sin(\pi r/2)^4 \lambda (1 + 16\lambda \sin(\pi r/2)^4)^{-1} \le 1/16$, it follows that

$$\lambda \delta_\lambda \le (1/16) \int_0^1 \cos(\pi r/2)^2 dr = 1/32.$$

Therefore,

$$\xi_{T\lambda} = 32\lambda(1-64\lambda\delta_{T\lambda})(1-64\lambda\delta_{T\lambda}+32^2\lambda^2(\delta_{T\lambda}^2-\eta_{T\lambda}^2))^{-1}+32^2\lambda^2(1-64\lambda\delta_{T\lambda}+32^2\lambda^2(\delta_{T\lambda}^2-\eta_{T\lambda}^2))^{-1}\delta_{T\lambda}$$

$$\to 32\lambda(1-64\lambda\delta_\lambda)(1-64\lambda\delta_\lambda+32^2\lambda^2\delta_\lambda^2)^{-1}+32^2\lambda^2(1-64\lambda\delta_\lambda+32^2\lambda^2\delta_\lambda^2)^{-1}\delta_\lambda$$

$$=\frac{32\lambda(1-64\lambda\delta_\lambda)+32^2\lambda^2\delta_\lambda}{(1-64\lambda\delta_\lambda+32^2\lambda^2\delta_\lambda^2)}=\frac{32\lambda}{1-32\lambda\delta_\lambda},$$

and

$$\phi_{T\lambda}=32^2\lambda^2(1-64\lambda\delta_{T\lambda}+32^2\lambda^2(\delta_{T\lambda}^2-\eta_{T\lambda}^2))^{-1}\eta_{T\lambda}$$

$$\to 32^2\lambda^2(1-32\lambda\delta_\lambda)^{-2}\times 0=0.$$

□

Proof of Theorem 3: The results $\lim_{T\to\infty} f_{T\lambda}(m)=\int_0^1\cos(\pi rm)(1+16\lambda\sin(\pi r/2)^4)^{-1}dr$ and $\lim_{T\to\infty} g_{T\lambda}(m)=\sqrt{2}\int_0^1\cos(\pi r(m-1/2))\sin(\pi r/2)^2\cos(\pi r/2)(1+16\lambda\sin(\pi r/2)^4)^{-1}dr$ are direct consequences of Lemma 3. To show that both representations for $f_\lambda(\cdot)$ and $g_\lambda(\cdot)$ are identical, note that one can represent $f_\lambda(m)$ as

$$f_\lambda(m)=\int_0^1\frac{\cos(\pi rm)}{1+16\lambda\sin(\pi r/2)^4}dr=\int_0^1\frac{\mathrm{Re}(\exp(i\pi rm))q}{q+(1-\exp(i\pi r))^2(1-\exp(-i\pi r))^2}dr$$

$$=\mathrm{Re}\left(\frac{1}{2}\int_{-1}^1\frac{\exp(i\pi rm)q}{q+(1-\exp(i\pi r))^2(1-\exp(-i\pi r))^2}dr\right)$$

by setting $\cos(\pi rm)=\mathrm{Re}(\exp(i\pi rm))$, where $\mathrm{Re}(x)$ stands for the real part of the complex number x, $\sin(\pi r/2)^4=(1-\exp(i\pi r))^2(1-\exp(-i\pi r))^2/16$, and $q=1/\lambda$. Using the change of variable $z = \exp(i\pi r)$ and $dz = i\pi z dr$, we now obtain

$$f_\lambda(m) = \mathrm{Re}\left(\frac{q}{2i\pi}\oint \frac{z^{m-1}}{q + (1-z)^2(1-z^{-1})^2} dz\right).$$

In the above expression, the denominator is a fourth order polynomial that has roots

$$r_1 = \frac{(2i - \sqrt{q}) + \sqrt{q - 4i\sqrt{q}}}{2i}$$

$$r_2 = \frac{(2i - \sqrt{q}) - \sqrt{q - 4i\sqrt{q}}}{2i}$$

$$r_3 = \frac{(2i + \sqrt{q}) - \sqrt{q + 4i\sqrt{q}}}{2i}$$

$$r_4 = \frac{(2i + \sqrt{q}) + \sqrt{q + 4i\sqrt{q}}}{2i}$$

These roots have the following relations: $r_2 = r_1^{-1}$, $r_4 = r_3^{-1}$, $r_3 = \bar{r}_1$, and $r_4 = \bar{r}_2$. Note that r_1 and r_3 are inside the unit circle, whereas r_2 and r_4 are outside the unit circle. Therefore,

$$f_\lambda(m) = \mathrm{Re}\left(\frac{q}{2i\pi}\oint \frac{z^{m-1}}{(z-r_1)(z-r_1^{-1})(z-\bar{r}_1)(z-\bar{r}_1^{-1})} dz\right).$$

Using Cauchy's Residue Theorem, the solution to this contour integral is

$$f_\lambda(m) = q\ \mathrm{Re}\left(\frac{r_1^{m+1}}{(r_1 - r_1^{-1})(r_1 - \bar{r}_1)(r_1 - \bar{r}_1^{-1})} + \frac{\bar{r}_1^{m+1}}{(\bar{r}_1 - r_1)(\bar{r}_1 - r_1^{-1})(\bar{r}_1 - \bar{r}_1^{-1})}\right).$$

By simplifying the above result, we can express $f_\lambda(m)$ as:

$$f_\lambda(m) = \frac{2q|r_1|^{m+2}\sin(\theta)(|r_1|^2\sin((m-1)\theta) - \sin((m+1)\theta))}{(1-2\cos(2\theta)|r_1|^2 + |r_1|^4)(|r_1|^2-1)(1-\cos(2\theta))}$$

where

$$\theta = \tan^{-1}\left(\frac{\sqrt{q-4i\sqrt{q}} + \sqrt{q+4i\sqrt{q}} - 2\sqrt{q}}{i(4i + \sqrt{q-4i\sqrt{q}} - \sqrt{q+4i\sqrt{q}})}\right).$$

Furthermore, one can also solve $g_\lambda(m)$ for $m \geq 1$ using the above approach, and this then gives

$$g_\lambda(m) = \sqrt{2}\int_0^1 \frac{\cos(\pi r(m-1/2))\sin(\pi r/2)^2\cos(\pi r/2)}{1+16\lambda\sin(\pi r/2)^4}dr.$$

In order to obtain our formula that relates $f_\lambda(m)$ to $g_\lambda(m)$, note that

$$\cos(\pi r(m-1/2))\sin(\pi r/2)^2\cos(\pi r/2) = \frac{1}{8}(\cos(\pi r(m-1)) - \cos(\pi r(m-2)) + \cos(\pi rm) - \cos(\pi r(m+1))),$$

which implies that $g_\lambda(m)$ can be written as

$$g_\lambda(m) = \frac{\sqrt{2}}{8}\int_0^1 \frac{\cos(\pi r(m-1)) - \cos(\pi r(m-2)) + \cos(\pi rm) - \cos(\pi r(m+1))}{1+16\lambda\sin(\pi r/2)^4}dr,$$

which in turn is equivalent to

$$g_\lambda(m) = (\sqrt{2}/8)\big(f_\lambda(m-1) - f_\lambda(m-2) + f_\lambda(m) - f_\lambda(m+1)\big)$$

for $m \geq 1$ by the definition of the function $f_\lambda(m)$. □

Proof of Theorem 4: First note that

$$E|(E|y_{[rT]}|)^{-1}\sum_{s=1}^{T} y_s(w_{T,[rT],s} - f_\lambda([rT]-s))|$$

$$\leq (E|y_{[rT]}|)^{-1}\sum_{s=1}^{T} E|y_s||w_{T,[rT],s} - f_\lambda([rT]-s))|$$

$$\leq \sup_{1\leq s\leq T}(E|y_s|(E|y_{[rT]}|)^{-1})\sum_{s=1}^{T}|w_{T,[rT],s} - f_\lambda([rT]-s)|,$$

and since by assumption,

$$\limsup_{T\to\infty}\sup_{1\leq s\leq T}(E|y_s|(E|y_{[rT]}|)^{-1}) < \infty,$$

it suffices to show

$$\limsup_{T\to\infty}\sum_{s=1}^{T}|f_{T\lambda}([rT]-s) - f_\lambda([rT]-s)| = 0$$

and

$$\limsup_{T\to\infty}\sum_{s=1}^{T}\sum_{j=2}^{8}|w^j_{T,[rT],s}| = 0.$$

The second result follows from the result of Equation (18). To show the first result, note that

$$\limsup_{T\to\infty}\sum_{s=1}^{T}|f_{T\lambda}([rT]-s)-f_{\lambda}([rT]-s)|$$

$$=\limsup_{T\to\infty}\sum_{s=0}^{[rT]-1}|f_{T\lambda}(s)-f_{\lambda}(s)|+\limsup_{T\to\infty}\sum_{s=1}^{T-[rT]}|f_{T\lambda}(s)-f_{\lambda}(s)|$$

and both terms vanish by the dominated convergence theorem because for $s\neq t$, by Theorem 1,

$$|f_{T\lambda}(t-s)-f_{\lambda}(t-s)|\leq C|t-s|^{-3}.$$

This now completes the argument. □

Proof of Theorem 5: Note that, by the properties $\sum_{s=1}^{T} w_{Tts}=1$ and $\sum_{s=1}^{T} w_{Tts}s=t$ that were noted in Section 2,

$$\hat{c}_{Tt}=y_t-\sum_{s=1}^{T} w_{Tts}(\alpha_1+\alpha_2 s+\alpha_3 z_s+u_s)$$

$$=u_t-\sum_{s=1}^{T} w_{Tts}u_s+\alpha_3(z_t-\sum_{s=1}^{T} w_{Tts}z_s)$$

$$= u_t - \sum_{s=1}^{T} w_{Tts} u_s + \alpha_3 (\sum_{j=1}^{t} \varepsilon_j - \sum_{s=1}^{T} \sum_{j=1}^{s} \varepsilon_j w_{Tts})$$

$$= u_t - \sum_{s=1}^{T} w_{Tts} u_s - \alpha_3 \sum_{j=1}^{T} \varepsilon_j (\sum_{s=j}^{T} w_{Tts} - I(j \leq t)).$$

Since $\hat{c}_{Tt} = \hat{c}_{Tt}^m$ for $m \geq T-1$ and because $\hat{c}_{Tt}^{-1} = u_t$ and $\sup_{t \geq 1} \| u_t \|_p < \infty$, it suffices to show that for every $m \geq -1$,

$$\sup_{T:T \geq m+2} \sup_{t \in [\gamma T, (1-\gamma)T]} \| \hat{c}_{Tt} - \hat{c}_{Tt}^m \|_p \leq C_1 (m+2)^{-1}$$

and therefore it also suffices to show that for every $m \geq 0$,

$$\sup_{T:T \geq m+1} \sup_{t \in [\gamma T, (1-\gamma)T]} \| \hat{c}_{Tt} - \hat{c}_{Tt}^{m-1} \|_p \leq C_1 (m+1)^{-1}.$$

Therefore, it suffices to show that

$$\sup_{T:T \geq m+1} \sup_{t \in [\gamma T, (1-\gamma)T]} \| \sum_{s=1}^{T} w_{Tts} u_s I(|t-s| \geq m) \|_p \leq C_2 (m+1)^{-2}$$

and

$$\sup_{T:T \geq m+1} \sup_{t \in [\gamma T, (1-\gamma)T]} \| \sum_{j=1}^{T} \varepsilon_j (\sum_{s=j}^{T} w_{Tts} - I(j \leq t)) I(|j-t| \geq m) \|_p \leq C_3 (m+1)^{-1}.$$

Because of the result of Equation (18), it follows that

$$\sup_{r \in [\gamma, 1-\gamma]} \sup_{T:T \geq m+1} T^3 \sup_{1 \leq s \leq T} \sum_{k=2}^{8} |w_{T,[rT],s}^k| < \infty.$$

To see the first result, note now that for $m \geq 0$

$$\sup_{T:T\geq m+1}\sup_{t\in[\gamma T,(1-\gamma)T]} \| \sum_{s=1}^{T} w_{Tts}u_s I(|t-s|\geq m) \|_p$$

$$\leq \sup_{T:T\geq m+1}\sup_{t\in[\gamma T,(1-\gamma)T]} \sum_{k=2}^{8}\sum_{s=1}^{T} |w_{Tts}^k| \sup_{s\geq 1} \| u_s \|_p$$

$$+ \sup_{T:T\geq m+1}\sup_{t\in[\gamma T,(1-\gamma)T]} \sum_{s=1}^{T} |f_{T\lambda}(|t-s|)| I(|t-s|\geq m) \sup_{s\geq 1} \| u_s \|_p$$

$$\leq (m+1)^{-2} \sup_{r\in[\gamma,1-\gamma]}\sup_{T:T\geq m+1} T^3 \sup_{1\leq s\leq T} \sum_{k=2}^{8} |w_{T,[rT],s}^k| \sup_{s\geq 1} \| u_s \|_p$$

$$+(2\sum_{k=0}^{\infty} Ck^{-3} I(k\neq 0) + I(k=0)) I(|k|\geq m) \sup_{s\geq 1} \| u_s \|_p$$

$$\leq C_3(m+1)^{-2} + C_4 I(m>0) \sum_{k=m}^{\infty} k^{-3} + C_5 I(m=0) \leq C_2(m+1)^{-2}.$$

For the second result, note that

$$z_t - \sum_{s=1}^{T} z_s w_{Tts}$$

$$= \sum_{j=1}^{t} \varepsilon_j - \sum_{s=1}^{T}\sum_{j=1}^{s} \varepsilon_j w_{Tts} = -\sum_{j=1}^{T} \varepsilon_j (\sum_{s=j}^{T} w_{Tts} - I(j\leq t)),$$

and therefore the second result follows because, using that $\sum_{s=1}^{T} w_{Tts} = 1$, defining sums over empty index sets as 0,

$$\| \sum_{j=1}^{T} \varepsilon_j (\sum_{s=j}^{T} w_{Tts} - I(j \leq t)) I(|j-t| \geq m) \|_p$$

$$\leq \sum_{j=1}^{t} I(t-j \geq m) | \sum_{s=j}^{T} w_{Tts} - 1 | \sup_{j \geq 1} \| \varepsilon_j \|_p$$

$$+ \sum_{j=t+1}^{T} I(j-t \geq m) | \sum_{s=j}^{T} w_{Tts} | \sup_{j \geq 1} \| \varepsilon_j \|_p$$

$$\leq \sum_{j=2}^{t} I(t-j \geq m) \sum_{s=1}^{j-1} |f_{T\lambda}(t-s)| \sup_{j \geq 1} \| \varepsilon_j \|_p$$

$$+ \sum_{j=t+1}^{T} I(j-t \geq m) \sum_{s=j}^{T} |f_{T\lambda}(t-s)| \sup_{j \geq 1} \| \varepsilon_j \|_p$$

$$+ \sum_{j=2}^{t} I(t-j \geq m) \sum_{k=2}^{8} \sum_{s=1}^{j-1} |w_{Tts}^k| \sup_{j \geq 1} \| \varepsilon_j \|_p$$

$$+ \sum_{j=t+1}^{T} I(j-t \geq m) \sum_{k=2}^{8} \sum_{s=j}^{T} |w_{Tts}^k| \sup_{j \geq 1} \| \varepsilon_j \|_p \, .$$

The first summation, divided by $\sup_{j \geq 1} \| \varepsilon_j \|_p$, is bounded by

$$\sum_{j=2}^{t} I(t-j \geq m) \sum_{s=1}^{j-1} C_6 (t-s)^{-3} = \sum_{s=1}^{t-1} \sum_{j=2}^{t} I(t-j \geq m) I(s \leq j-1) C_6 (t-s)^{-3}$$

$$\leq \sum_{s=1}^{t-1} C_6 I(s \leq t-m-1)(t-s)^{-2} \leq C_6 \sum_{k=m+1}^{\infty} k^{-2} \leq C_7 (m+1)^{-1}.$$

For the second term, a similar argument holds. For the third term, note that

$$\sup_{T:T\geq m+1} \sup_{t\in[\gamma T,(1-\gamma)T]} \sum_{j=2}^{t} I(t-j\geq m) \sum_{k=2}^{8} \sum_{s=1}^{j-1} |w_{Tts}^{k}| \sup_{j\geq 1} \parallel \varepsilon_j \parallel_p$$

$$\leq (m+1)^{-1} \sup_{T:T\geq m} \sup_{r\in[\gamma,1-\gamma]} T^3 \sup_{1\leq s\leq T} \sum_{k=2}^{8} |w_{T,[rT],s}^{k}| \sup_{j\geq 1} \parallel \varepsilon_j \parallel_p$$

$$\leq C_8(m+1)^{-1}$$

and a similar argument holds for the fourth term. Therefore, the conclusion of the theorem now follows. □

Proof of Theorem 6: Write

$$T^{-1}\sum_{t=1}^{T}(f(\hat{c}_{Tt}) - Ef(\hat{c}_{Tt}))$$

$$= T^{-1} \sum_{t\in\{1,\ldots,T\},t\notin[\gamma T,(1-\gamma)T]} (f(\hat{c}_{Tt}) - Ef(\hat{c}_{Tt})),$$

$$+T^{-1} \sum_{t\in\{1,\ldots,T\},t\in[\gamma T,(1-\gamma)T]} (f(\hat{c}_{Tt}) - Ef(\hat{c}_{Tt}))$$

and note that the first term is bounded in absolute value by

$$4\gamma \sup_{x\in\mathbb{R}} |f(x)|,$$

and because γ can be chosen arbitrarily small, it therefore suffices to show a weak law of large numbers for the second term. To show this, first note that for all $K > 0$ and $\eta > 0$,

$$\sup_{T\geq 1, t\in[\gamma T,(1-\gamma)T]} \| f(\hat{c}_{Tt}) - E(f(\hat{c}_{Tt})|v_{t-m},\ldots,v_{t+m}) \|_2$$

$$\leq \sup_{T\geq 1, t\in[\gamma T,(1-\gamma)T]} \| (f(\hat{c}_{Tt}) - E(f(\hat{c}_{Tt})|v_{t-m},\ldots,v_{t+m}))I(|\hat{c}_{Tt}| > K) \|_2$$

$$+ \sup_{T\geq 1, t\in[\gamma T,(1-\gamma)T]} \| (f(\hat{c}_{Tt}) - E(f(\hat{c}_{Tt})|v_{t-m},\ldots,v_{t+m}))I(|\hat{c}_{Tt}| \leq K)I(|\hat{c}_{Tt} - \hat{c}^m_{Tt}| > \eta) \|_2$$

$$+ \sup_{T\geq 1, t\in[\gamma T,(1-\gamma)T]} \| (f(\hat{c}_{Tt}) - E(f(\hat{c}_{Tt})|v_{t-m},\ldots,v_{t+m}))I(|\hat{c}_{Tt}| \leq K)I(|\hat{c}_{Tt} - \hat{c}^m_{Tt}| \leq \eta) \|_2 .$$

Because $f(\cdot)$ is bounded in absolute value,

$$\sup_{T\geq 1, t\in[\gamma T,(1-\gamma)T]} \| (f(\hat{c}_{Tt}) - E(f(\hat{c}_{Tt})|v_{t-m},\ldots,v_{t+m}))I(|\hat{c}_{Tt}| > K) \|_2^2$$

$$\leq 2\sup_{x\in\mathbb{R}}|f(x)| \sup_{T\geq 1, t\in[\gamma T,(1-\gamma)T]} P(|\hat{c}_{Tt}| > K)$$

$$\leq 2\sup_{x\in\mathbb{R}}|f(x)|K^{-2} \sup_{T\geq 1, t\in[\gamma T,(1-\gamma)T]} E|\hat{c}_{Tt}|^2,$$

while similarly,

$$\sup_{T\geq 1, t\in[\gamma T,(1-\gamma)T]} \| (f(\hat{c}_{Tt}) - E(f(\hat{c}_{Tt})|v_{t-m},\ldots,v_{t+m}))I(|\hat{c}_{Tt}| \leq K)I(|\hat{c}_{Tt} - \hat{c}^m_{Tt}| > \eta) \|_2^2$$

$$\leq 2\sup_{x\in\mathbb{R}}|f(x)|\eta^{-2} \sup_{T\geq 1, t\in[\gamma T,(1-\gamma)T]} E|\hat{c}_{Tt} - \hat{c}^m_{Tt}|^2,$$

and

$$\sup_{T\geq 1, t\in[\gamma T,(1-\gamma)T]} \| (f(\hat{c}_{Tt}) - E(f(\hat{c}_{Tt})|v_{t-m},\ldots,v_{t+m}))I(|\hat{c}_{Tt}|\leq K)I(|\hat{c}_{Tt}-\hat{c}^{m}_{Tt}|\leq\eta) \|_2$$

$$\leq \sup_{|x|\leq K}\ \sup_{x'\in\mathbb{R}:|x-x'|\leq\eta} |f(x)-f(x')|.$$

Therefore, by making first m approach infinity, then making η approach 0, and then K approach infinity, it now follows that

$$\lim_{m\to\infty}\ \sup_{T\geq 1, t\in[\gamma T,(1-\gamma)T]} \| f(\hat{c}_{Tt}) - E(f(\hat{c}_{Tt})|v_{t-m},\ldots,v_{t+m}) \|_2 = 0.$$

Therefore, $f(\hat{c}_{Tt})$ is near epoch dependent on v_t for $t\in[\gamma T,(1-\gamma)T]$, and it therefore is a bounded L_1-mixingale as defined in Andrews (1988). Therefore, Andrews' weak law of large numbers applies, and the proof of the theorem is complete. □

Proof of Theorem 7: Since $f_\lambda(m) = \int_0^1 \cos(\pi rm)h_\lambda(r)dr$ where $h_\lambda(r) = l(2\lambda^{1/4}\sin(\pi r/2))$ for $l(x) = (1+x^4)^{-1}$, it follows by partial integration that for any integer m

$$f_\lambda(m) = (\pi m)^{-1}\int_0^1 h_\lambda(r)d\sin(\pi rm) = -(\pi m)^{-1}\int_0^1 \sin(\pi rm)h'_\lambda(r)dr$$

$$= (\pi m)^{-2}\int_0^1\ h'_\lambda(r)d\cos(\pi rm)$$

$$= \left[(\pi m)^{-2}\ h'_\lambda(r)\cos(\pi rm)\right]_0^1 - (\pi m)^{-2}\int_0^1 \cos(\pi rm)dh'_\lambda(r).$$

Now

$$h'_\lambda(r) = l'(2\lambda^{1/4}\sin(\pi r/2))2\lambda^{1/4}\cos(\pi r/2))(\pi/2),$$

and therefore

$$\left[(\pi m)^{-2}\ h'_\lambda(r)\cos(\pi r m)\right]_0^1 = 0,$$

implying that

$$|f_\lambda(m)| \leq (\pi m)^{-2}\int_0^1 |h''_\lambda(r)|dr.$$

Furthermore,

$$h''_\lambda(r) = l''(2\lambda^{1/4}\sin(\pi r/2))(2\lambda^{1/4}\cos(\pi r/2))(\pi/2))^2 - l'(2\lambda^{1/4}\sin(\pi r/2))2\lambda^{1/4}\sin(\pi r/2)(\pi/2)^2,$$

and

$$\int_0^1 |l'(2\lambda^{1/4}\sin(\pi r/2))2\lambda^{1/4}\sin(\pi r/2)(\pi/2)^2|dr$$

$$\leq (\pi^2/2)\lambda^{1/4}\sup_{x\geq 0}|l'(x)|,$$

and therefore it only remains to show that

$$\int_0^1 |l''(2\lambda^{1/4}\sin(\pi r/2))(2\lambda^{1/4}\cos(\pi r/2))(\pi/2))^2|dr \leq C_1\lambda^{1/4}.$$

Since

$$|l''(x)| = |4x^2(5x^4-3)(x^4+1)^{-3}| \leq (x^4+1)^{-3/2} \sup_{x\in\mathbb{R}} |4x^2(5x^4-3)(x^4+1)^{-3/2}|$$

$$\leq C_2(x^4+1)^{-3/2}$$

and because $\sin(\pi r/2) \geq r$ for $r \in [0,1]$, it now follows that

$$\int_0^1 |l''(2\lambda^{1/4}\sin(\pi r/2))(2\lambda^{1/4}\cos(\pi r/2))(\pi/2))^2|dr$$

$$\leq C_2 \int_0^1 (1+(2\lambda^{1/4}\sin(\pi r/2))^4)^{-3/2}(2\lambda^{1/4}\cos(\pi r/2))(\pi/2))^2 dr$$

$$\leq C_2\lambda^{1/4} \int_0^1 (1+(2\lambda^{1/4}r))^4)^{-3/2}\lambda^{1/4}dr$$

$$= C_2\lambda^{1/4} \int_0^{\lambda^{1/4}} (1+(2s)^4)^{-3/2}ds$$

$$\leq C_2\lambda^{1/4} \int_0^{\infty} (1+(2s)^4)^{-3/2}ds,$$

and therefore,

$$|f_\lambda(m)| \leq C_3 m^{-2}\lambda^{1/4}.$$

From this it follows that

$$\lambda^{1/4}|f_\lambda(\lambda^{1/4}m)| \leq C_3\lambda^{1/4}\lambda^{1/4}(\lambda^{1/4}m)^{-2} = C_3 m^{-2}.$$

Also, for any $m \in \mathbb{R}$,

$$\lambda^{1/4} f_\lambda(\lambda^{1/4} m) = \lambda^{1/4} \int_0^1 \cos(\pi r \lambda^{1/4} m) h_\lambda(r) dr = \int_0^{\lambda^{1/4}} \cos(\pi y m) h_\lambda(\lambda^{-1/4} y) dy$$

$$= \int_0^\infty I(y \le \lambda^{1/4}) \cos(\pi y m) l(2\lambda^{1/4} \sin(\pi y/(2\lambda^{1/4}))) dy$$

and because $\sin(\pi x/2) \ge x$ for $x \in [0,1]$,

$$|I(y \le \lambda^{1/4}) \cos(\pi y m) l(2\lambda^{1/4} \sin((1/2)\pi y \lambda^{-1/4}))| \le |l(2y)|$$

and $\int_0^\infty |l(2y)| dy < \infty$. Therefore, by the dominated convergence theorem,

$$\lim_{\lambda\to\infty} \lambda^{1/4} f_\lambda(\lambda^{1/4} m)$$

$$= \int_0^\infty \lim_{\lambda\to\infty} I(y \le \lambda^{1/4}) \cos(\pi y m) l(2\lambda^{1/4} \sin((1/2)\pi y \lambda^{-1/4})) dy$$

$$= \int_0^\infty \cos(\pi y m) l(\pi y) dy = f(m).$$

Similarly,

$$\lambda^{3/4} g_\lambda(\lambda^{1/4} m)$$

$$= \sqrt{2} \int_0^\infty I(y \le \lambda^{1/4}) \cos(\pi y m - \pi y/(2\lambda^{1/4})) \lambda^{1/2} \sin(\pi y/(2\lambda^{1/4}))^2 \cos(\pi r/(2\lambda^{1/4})) l(2\lambda^{1/4} \sin(\pi y/(2\lambda^{1/4}))) dy$$

and, noting that $\lambda^{1/2}\sin(\pi y/(2\lambda^{1/4}))^2 \leq (1/4)\pi^2 y^2$ for $y \in \mathbb{R}$,

$$|I(y \leq \lambda^{1/4})\cos(\pi y m - \pi y/(2\lambda^{1/4}))\lambda^{1/2}\sin(\pi y/(2\lambda^{1/4}))^2\cos(\pi r/(2\lambda^{1/4}))l(2\lambda^{1/4}\sin(\pi r/(2\lambda^{1/4})))|$$

$$\leq (1/4)\pi^2 y^2 l(2y)$$

and $\int_0^\infty (1/4)\pi^2 y^2 |l(2y)| dy < \infty$. Therefore,

$$\lim_{\lambda\to\infty}\lambda^{3/4} g_\lambda(\lambda^{1/4} m)$$

$$= \sqrt{2}\int_0^\infty \cos(\pi y m)(\lim_{\lambda\to\infty}\lambda^{1/2}\sin(\pi y/(2\lambda^{1/4}))^2) l(\pi y) dy$$

$$= \sqrt{2}\int_0^\infty \cos(\pi y m)(\pi y/2)^2 l(\pi y) dy = g(m).$$

For showing the uniform convergence result, it now suffices to show equicontinuity on $[0, K]$, viz.

$$\limsup_{\eta\downarrow 0}\limsup_{\lambda\to\infty}\sup_{|m|\leq K, |m-m'|\leq\eta}|\lambda^{1/4} f_\lambda(\lambda^{1/4} m) - \lambda^{1/4} f_\lambda(\lambda^{1/4} m')| = 0.$$

To show this, note that

$$\lambda^{1/4} f_\lambda(\lambda^{1/4} m) = \lambda^{1/4}\int_0^1 \cos(m r \pi \lambda^{1/4}) h_\lambda(r) dr = \int_0^{\lambda^{1/4}} \cos(m y \pi) l(2\lambda^{1/4}\sin(\pi\lambda^{-1/4} y/2)) dy,$$

so, because $l(x)$ is nonincreasing on $[0,\infty)$ and $\sin(\pi x/2) \geq x$ for $x \in [0,1]$,

$$\sup_{|m|\leq K, |m-m'|\leq\eta} |\lambda^{1/4} f_\lambda(\lambda^{1/4}m) - \lambda^{1/4} f_\lambda(\lambda^{1/4}m')|$$

$$\leq \int_0^{\lambda^{1/4}} \sup_{m,|m-m'|\leq\eta} |\cos(my\pi) - \cos(m'y\pi)| l(2\lambda^{1/4}\sin(\pi\lambda^{-1/4}y/2))dy$$
$$\leq \int_0^{\infty} \sup_{m,|m-m'|\leq\eta} |\cos(my\pi) - \cos(m'y\pi)| l(2y)dy,$$

and by the dominated convergence theorem, using the integrability of $|l(2y)|$, it now follows that

$$\limsup_{\eta\downarrow 0} \limsup_{\lambda\to\infty} \sup_{|m|\leq K, |m-m'|\leq\eta} |\lambda^{1/4} f_\lambda(\lambda^{1/4}m) - \lambda^{1/4} f_\lambda(\lambda^{1/4}m')| = 0.$$

□

Proof of Theorem 8: Because of the result of Theorem 4, it suffices to show that

$$(E|y_{[rT]}|)^{-1} E|\sum_{s=1}^{T} y_s (f_\lambda([rT]-s) - \lambda^{-1/4} f(\lambda^{-1/4}([rT]-s)))| \to 0.$$

Similarly to the proof of Theorem 4, for all $K > 0$,

$$\limsup_{T\to\infty} (E|y_{[rT]}|)^{-1} E|\sum_{s=1}^{T} y_s (f_\lambda([rT]-s) - \lambda^{-1/4} f(\lambda^{-1/4}([rT]-s)))|$$

$$\leq \limsup_{T\to\infty} \max_{1\leq s\leq T} E|y_s|(E|y_{[rT]}|)^{-1} \sum_{s=-\infty}^{\infty} |f_\lambda([rT]-s) - \lambda^{-1/4} f(\lambda^{-1/4}([rT]-s))|$$

$$\leq C_1 \sum_{m=0}^{\infty} |f_\lambda(m) - \lambda^{-1/4} f(\lambda^{-1/4} m)|$$

$$\leq C_1 \sum_{m=0}^{[K\lambda^{1/4}]} |f_\lambda(m) - \lambda^{-1/4} f(\lambda^{-1/4} m)| + C_1 \sum_{m=[K\lambda^{1/4}]+1}^{\infty} |f_\lambda(m)| + C_1 \sum_{m=[K\lambda^{1/4}]+1}^{\infty} \lambda^{-1/4} |f(\lambda^{-1/4} m)|.$$

By Theorem 7,

$$\limsup_{\lambda\to\infty} \sum_{m=0}^{[K\lambda^{1/4}]} |f_\lambda(m) - \lambda^{-1/4} f(\lambda^{-1/4} m)|$$

$$\leq K \limsup_{\lambda\to\infty} \lambda^{1/4} \sup_{m=0,1,\ldots,[K\lambda^{1/4}]} |f_\lambda(m) - \lambda^{-1/4} f(\lambda^{-1/4} m)|$$

$$\leq K \limsup_{\lambda\to\infty} \sup_{0\leq y\leq K} |\lambda^{1/4} f_\lambda(\lambda^{1/4} y) - f(y)| = 0.$$

Also, because $|f_\lambda(m)| \leq C_2 \lambda^{1/4} m^{-2}$ for $m \geq 1$ by Theorem 7,

$$\sum_{m=[K\lambda^{1/4}]+1}^{\infty} |f_\lambda(m)| \leq C_2 \lambda^{1/4} \sum_{m=[K\lambda^{1/4}]+1}^{\infty} m^{-2} \leq C_2 K^{-1}$$

and because for $m \geq 1$, $|f(m)| \leq C_3 m^{-2}$,

$$\sum_{m=[K\lambda^{1/4}]+1}^{\infty} \lambda^{-1/4} |f(\lambda^{-1/4} m)| \leq C_3 \sum_{m=[K\lambda^{1/4}]+1}^{\infty} \lambda^{1/4} m^{-2} \leq C_4 K^{-1}$$

and since K was arbitrary, the result now follows. □

Proof of Theorem 9: Note that

$$\limsup_{T\to\infty}(E|y_{[4rT]}|)^{-1}E|\hat{\tau}_{4T,4[rT]-4+q}(\lambda,\{y_s,s=1,\ldots,4T\})-\hat{\tau}_{4T,[4Tr]}(\lambda,\{y_s,s=1,\ldots,4T\})|$$

$$\leq \limsup_{T\to\infty}\max_{1\leq s\leq 4T}E|y_s|(E|y_{[4rT]}|)^{-1}\sum_{s=1}^{4T}|w_{4T,4[rT]-4+q,s}-w_{4T,[4Tr],s}|$$

$$\leq C_1\limsup_{T\to\infty}\sum_{s=1}^{4T}|f_{4T,\lambda}(4[rT]-4+q-s)-f_{4T,\lambda}([4Tr]-s)|$$

$$+C_1\limsup_{T\to\infty}\sum_{s=1}^{4T}\sum_{k=2}^{8}|w^k_{4T,4[rT]-4+q,s}|+C_1\limsup_{T\to\infty}\sum_{s=1}^{4T}\sum_{k=2}^{8}|w^k_{4T,[4Tr],s}|.$$

The result of Equation (18) ensures that the third term is $O(T^{-2})$. Using the definition of the w^k_{Tts} for $k=2,..,8$, it is easy to see that the second term is also $O(T^{-2})$. Because

$$|4[rT]-4+q-[4rT]|\leq 7,$$

the first term is bounded by

$$\limsup_{T\to\infty}\sum_{m=-\infty}^{\infty}I(|m|\leq 4T-1)I(|m+l|\leq 4T-1)\max_{|l|\leq 7}|f_{4T,\lambda}(m)-f_{4T,\lambda}(m+l)|$$

and therefore it suffices to show that for all l, $|l|\leq 7$,

$$\limsup_{\lambda\to\infty}\limsup_{T\to\infty}\sum_{m=-\infty}^{\infty}I(|m|\leq 4T-1)I(|m+l|\leq 4T-1)|f_{4T,\lambda}(m)-f_{4T,\lambda}(m+l)|$$

equals 0. By the upper bound on $f_{T,\lambda}(m)$ of Theorem 1 and the dominated convergence theorem, the last expression equals

$$\limsup_{\lambda\to\infty}\sum_{m=-\infty}^{\infty}|f_\lambda(m)-f_\lambda(m+l)|$$

and is therefore bounded by

$$\limsup_{\lambda\to\infty}\sum_{m=-\infty}^{\infty}|\lambda^{-1/4}f(\lambda^{-1/4}m)-\lambda^{-1/4}f(\lambda^{-1/4}(m+l))|$$

$$+4\limsup_{\lambda\to\infty}\sum_{m=0}^{\infty}|f_\lambda(m)-\lambda^{-1/4}f(\lambda^{-1/4}m)|,$$

and the second term was shown to equal 0 in the proof of Theorem 8. For the first term, note that we can write

$$\limsup_{\lambda\to\infty}\sum_{m=-\infty}^{\infty}|\lambda^{-1/4}f(\lambda^{-1/4}m)-\lambda^{-1/4}f(\lambda^{-1/4}(m+l))|$$

$$=\limsup_{\lambda\to\infty}\int_{-\infty}^{\infty}|\lambda^{-1/4}f(\lambda^{-1/4}[m])-\lambda^{-1/4}f(\lambda^{-1/4}([m]+l))|dm$$
$$=\limsup_{\lambda\to\infty}\int_{-\infty}^{\infty}f(\lambda^{-1/4}[\lambda^{1/4}y])-f(\lambda^{-1/4}([\lambda^{1/4}y]+l))|dy$$

and the last term equals 0 by the dominated convergence theorem because $\sup_{m\in\mathbb{R}}|f(m)|(m+1)^2<\infty$ and the continuity of $f(\cdot)$.

Now by Theorem 8,

$$\limsup_{\lambda\to\infty}\limsup_{T\to\infty}(E|y_{[4rT]}|)^{-1}E|\hat{\tau}_{4T,[4rT]}(\lambda,\{y_s,s=1,\ldots,4T\})-\bar{\tau}_{4T,[4rT]}(\lambda,\{y_s,s=1,\ldots,4T\})|=0$$

and

$$\limsup_{\lambda\to\infty}\limsup_{T\to\infty}(E|y_{[rT]}|)^{-1}E|\hat{\tau}_{T,[rT]}(\lambda,\{(1/4)\sum_{q=1}^{4}y_{4i-4+q}:i=1,\ldots,T\})$$

$$-\bar{\tau}_{T,[rT]}(\lambda,\{(1/4)\sum_{q=1}^{4}y_{4i-4+q}:i=1,\ldots,T\})|=0$$

because the regularity condition of Theorem 8 was assumed in the statement of the Theorem.

Since

$$\bar{\tau}_{T,[rT]}(\lambda,\{z_s:s=1,...,T\})=\sum_{s=1}^{T}z_s\lambda^{-1/4}f(\lambda^{-1/4}(t-s))$$

it follows that

$$\bar{\tau}_{4T,[4rT]}(\lambda,\{y_s,s=1,\ldots,4T\})=\sum_{s=1}^{4T}y_s\lambda^{-1/4}f(\lambda^{-1/4}([4rT]-s))$$

and

$$\bar{\tau}_{T,[rT]}(4^{-4}\lambda,\{(1/4)\sum_{q=1}^{4}y_{4i-4+q}:i=1,\ldots,T\})$$

$$=\sum_{i=1}^{T}(1/4)\sum_{q=1}^{4}y_{4i-4+q}(4^{-4}\lambda)^{-1/4}f((4^{-4}\lambda)^{-1/4}([rT]-i))$$

$$=\sum_{i=1}^{T}\sum_{q=1}^{4}y_{4i-4+q}\lambda^{-1/4}f(\lambda^{-1/4}4([rT]-i))$$

$$= \sum_{s=1}^{4T} y_s \lambda^{-1/4} f(\lambda^{-1/4} 4([rT] - ([(s-1)/4] + 1))).$$

Since

$$|([4rT] - s) - 4([rT] - ([(s-1)/4] + 1))| \leq 7,$$

it follows that

$$(E|y_{[4rT]}|)^{-1} E|\sum_{s=1}^{4T} y_s \lambda^{-1/4} f(\lambda^{-1/4}([4rT] - s)) - \lambda^{-1/4} f(\lambda^{-1/4} 4([rT] - ([(s-1)/4] + 1)))|$$

$$\leq C_2 \sum_{m=-\infty}^{\infty} \max_{|l| \leq 7} |\lambda^{-1/4} f(\lambda^{-1/4} m) - \lambda_Q^{-1/4} f(\lambda^{-1/4}(m + l))|,$$

and this expression was earlier shown to approach 0 as $\lambda \to \infty$. □