Negative Powers of Integrated Processes

Neslihan Sakarya[*] Robert M. de Jong[†]

March 2, 2021

Abstract

This paper derives the limit distribution of the rescaled sum of the absolute value of an integrated process with continuously distributed innovations raised to a negative power less than -1, and of the analogous statistic that is obtained using the same function of an integrated process but only considering positive values of the integrated process. We show that the limit behavior of this statistic is determined by the values of the integrated process that are closest to 0, and find the limit behavior of the values of the integrated process that are closest to 0.

1 Introduction

Pötscher (2013) studied statistics of the form $\sum_{t=1}^n |x_t|^{-q}$ for $q > 1$ and an integrated process x_t satisfying some regularity conditions, and established the order of magnitude of such statistics. Among other results, Pötscher (2013) showed that $n^{-q/2} \sum_{t=1}^n |x_t|^{-q} = O_p(1)$ for $q > 1$. This paper will show that $n^{-q/2} \sum_{t=1}^n |x_t|^{-q}$ converges in distribution for $q > 1$ under regularity conditions, and provides a characterization of the limit distribution. Therefore,

[*]Department of Economics, University of Essex, email ns17003@essex.ac.uk
[†]Department of Economics, Ohio State University, email de-jong.8@osu.edu

the rate established in Pötscher (2013) was indeed the best possible one. Note that in order to prevent a division by zero issue, a regularity condition ensuring the continuity of the distribution of x_t is needed to analyze such statistics.

As Pötscher (2013) noted, the asymptotic behavior of expressions of the form $r_n \sum_{t=1}^n f(k_n x_t)$, for deterministic sequences r_n and k_n, has been the subject of a number of articles in recent years. In Econometrics, Park and Phillips (1999) started the interest in this topic. The work of Pötscher (2004) and de Jong (2004) contained results that showed that for $r_n = n^{-1}$, $k_n = n^{-1/2}$, $n^{-1/2} x_{[rn]} \Rightarrow \lambda W(r)$ for $r \in [0,1]$, $W(\cdot)$ Brownian motion, λ^2 a variance parameter, and $f(\cdot)$ absolutely integrable on finite intervals, the limit $\int_0^1 f(\lambda W(r))dr$ can be found under regularity conditions. Therefore, because for $0 < q < 1$, $|x|^{-q}$ satisfies $\int_a^b |x|^{-q} dx < \infty$ for any $a, b \in \mathbb{R}$, $a \leq b$, it follows that $n^{-1+q/2} \sum_{t=1}^n |x_t|^{-q} \xrightarrow{d} \int_0^1 |\lambda W(r)|^{-q} dr$. This result does not follow immediately from the continuous mapping, because the pole at 0 renders the mapping non-continuous. Such results however are not informative about the case where $f(\cdot)$ has a non-integrable pole, such as $f(x) = |x|^{-q}$ for $q \geq 1$.

For the case $q = 1$, Pötscher (2013) showed that $n^{-1/2}(\log(n))^{-1} \sum_{t=1}^n |x_t|^{-1} = O_p(1)$ under regularity conditions, and Michel and de Jong (2020) showed that

$$n^{-1/2}(\log(n))^{-1} \sum_{t=1}^{n} |x_t|^{-1} \xrightarrow{d} 2\lambda^{-1}|Z| \tag{1}$$

under regularity conditions, where $Z \sim N(0,1)$.

Note that the above results all have one-sided equivalents that are obtained by only summing over the values of t for which x_t is positive or negative. Other papers considering statistics of the form $r_n \sum_{t=1}^n f(k_n x_t)$ are Borodin and Ibragimov (1995), Jeganathan (2004), de Jong and Wang (2005), Berkes and Horváth (2006), and Christopeit (2009).

The plan of this paper is as follows. Section 2 outlines the idea of the proof of the main result. In Section 3, we first set out to find convergence in distribution results for the occupation times for small intervals. Note that Akonom (1993, Théorème 2 and Lemme 1) established results for occupation times of the integrated process, but this author's results are not sufficiently tailored to the "small" interval situation to be of use here. We apply these results to show convergence results for $\min_{\{t:1\leq t\leq n, x_t>0\}} x_t$ and $\min_{1\leq t\leq n} |x_t|$. In Section 4, we consider multivariate convergence in distribution results for occupation times and convergence results for the order statistics of $|x_t|$ and the positive values of x_t. Section 5 then derives the limit distribution for the statistics of $n^{-q/2}\sum_{t=1}^n |x_t|^{-q}$ and $n^{-q/2}\sum_{t=1}^n x_t^{-q} I(x_t > 0)$ for $q > 1$. We conclude with Section 6, where we give simulation results for the distributions of $\min_{\{t:1\leq t\leq n, x_t>0\}} x_t$,$\min_{1\leq t\leq n} |x_t|$, $n^{-1}\sum_{t=1}^n x_t^{-2} I(x_t > 0)$, and $n^{-1}\sum_{t=1}^n x_t^{-2}$.

2 Main idea of the proof

Let x_t be an integrated process that is a recurrent random walk with i.i.d. innovations. In this paper, our approach is to write

$$n^{-q/2}\sum_{t=1}^n |x_t|^{-q} = \sum_{t=1}^n |n^{1/2} x_t|^{-q} = \sum_{t=1}^n Z_{nt}^{-q} \tag{2}$$

where Z_{nt} is the t-th smallest value of $n^{1/2}|x_t|$, $t = 1, \ldots, n$. We then show the joint convergence of $(Z_{n1}, \ldots, Z_{nm})'$ for any integer m, and prove that $n^{-q/2}\sum_{t=1}^n |x_t|^{-q}$ is asymptotically close to $\sum_{t=1}^m Z_{nt}^{-q}$. We find the limit distribution of $n^{-q/2}\sum_{t=1}^n |x_t|^{-q}$ based on those results. For $n^{-q/2}\sum_{t=1}^n x_t^{-q} I(x_t > 0)$, the reasoning is similar, by noting that

$$n^{-q/2}\sum_{t=1}^n x_t^{-q} I(x_t > 0) = \sum_{t=1}^n (n^{1/2} x_t)^{-q} I(x_t > 0) = \sum_{t=1}^{M_n} Y_{nt}^{-q} \tag{3}$$

where M_n is the number of positive x_t and Y_{nt} is the t-th smallest positive value of $n^{1/2}x_t$, $t = 1, \ldots, n$.

We will first focus on the case $m = 1$ and $Z_{n1} = n^{1/2} \min_{1\leq t\leq n} |x_t|$. Note that of course, from the functional central limit theorem it follows that

$$n^{-1/2} \min_{1\leq t\leq n} |x_t| \xrightarrow{d} \inf_{r\in[0,1]} |W(r)| = 0 \tag{4}$$

but beyond that, the functional central limit theorem is not informative about $\min_{1\leq t\leq n} |x_t|$.

The behavior of $n^{1/2} \min_{1\leq t\leq n} |x_t|$, but also more general order statistics of this type, can be related to occupation times of integrated processes for small intervals, where "small" here means that the width of the interval is $O(n^{-1/2})$. Define $S_n(y) = \sum_{t=1}^{n} I(|x_t| \leq yn^{-1/2})$. Then for all $y \in \mathbb{R}$,

$$P(Z_{n1} \leq y) = P(S_n(y) > 1/2). \tag{5}$$

If we can now show the convergence in distribution of $S_n(y)$ to some limit $S(y)$ for all $y \in \mathbb{R}$, by noting that $1/2$ is necessarily a continuity point of the distribution of $S(y)$ because the distribution of $S(y)$ will only put probability mass on the integers, we also find

$$\lim_{n\to\infty} P(Z_{n1} \leq y). \tag{6}$$

A similar argument can be used for $Y_{n1} = n^{1/2} \min_{\{t:1\leq t\leq n, x_t>0\}} x_t$. Defining $R_n(y) = \sum_{t=1}^{n} I(0 < x_t \leq yn^{-1/2})$ we can also note that for all $y \in \mathbb{R}$,

$$P(Y_{n1} \leq y) = P(R_n(y) > 1/2). \tag{7}$$

Therefore, the next section will first establish results on the limit behavior of the occupation times for small intervals.

3 Occupation times for small intervals

Let x_t be an integrated process. In this section we will show that $R_n(y)$ and $S_n(y)$ converge in distribution under regularity conditions. To show this, we first establish that all positive integer moments $E(R_n(y))^p$ and $E(S_n(y))^p$ converge. In the lemma below and everywhere in this paper, $F_t(\cdot)$ and $f_t(\cdot)$ denote the distribution function and density function of $t^{-1/2}x_t$, respectively (and therefore, the existence of $f_t(\cdot)$ is assumed for all t). Let $\phi(\cdot)$ denote the density function of the standard normal distribution. For the results of this paper, we need the following assumption:

Assumption 1. *Δx_t is i.i.d., $\sup_{x\in\mathbb{R}} |f_n(x) - \phi(x)| \to 0$, and $\sup_{t\geq 1, x\in\mathbb{R}} |f'_t(x)| < \infty$.*

Note that the condition $\sup_{x\in\mathbb{R}} |f_n(x) - \phi(x)| \to 0$ implies that a variance rescaling to 1 has been imposed.

Akonom (1993) considers the case where Δx_t has characteristic function $\psi(r)$ and is i.i.d., $x_0 = 0$, $E\Delta x_t = 0$, $E(\Delta x_t)^2 < \infty$. From the arguments in Akonom (1993, p.61-62), it follows that Assumption 1 is then implied by $E(\Delta x_t)^2 = 1$ and $\int_{-\infty}^{\infty} |r||\psi(r)|dr < \infty$. Akonom (1993) also shows that if for some $\beta > 0$, $\lim_{r\to\infty} |r|^\beta |\psi(r)| = 0$, then there exists an integer t^* such that $\sup_{t\geq t^*, x\in\mathbb{R}} |f'_t(x)| < \infty$. For this paper, we use Assumption 1, which avoids having to split up a number of summations in the proofs into the $t < t^*$ and $t \geq t^*$ cases, but there does not seem to be a fundamental difficulty with this generalization. Pötscher (2013) used boundedness of density conditions for $t \geq t^*$ for some t^*.

Let $\Gamma(\cdot)$ denote the gamma function and let $\Delta^k \mu_p$ denote the k-th difference of μ_p. We can show the convergence of all positive integer-valued moments of $R_n(y)$ and $S_n(y)$ and deduce convergence in distribution from that, giving the following result:

Lemma 1. *Under Assumption 1,*

1. *for all $y \in \mathbb{R}$, there exists a random variable $R(y)$ with moments μ_p satisfying $\mu_1 = y\sqrt{2/\pi}$ and, for $p \geq 2$,*

$$\Delta^{p-1}\mu_p = p! y^p 2^{-p/2} / \Gamma(p/2 + 1), \tag{8}$$

such that

$$R_n(y) \xrightarrow{d} R(y); \tag{9}$$

2. *for all $y \in \mathbb{R}$, there exists a random variable $S(y)$ with moments ν_p satisfying $\nu_1 = 2y\sqrt{2/\pi}$ and, for $p \geq 2$,*

$$\Delta^{p-1}\nu_p = p!y^p 2^{p/2}/\Gamma(p/2+1), \tag{10}$$

such that

$$S_n(y) \xrightarrow{d} S(y). \tag{11}$$

Note that the distributions of $R(y)$ and $S(y)$ do not depend on the distribution of Δx_t.

One might incorrectly conjecture, based on the standard FCLT plus continuous mapping theorem reasoning and the occupation times formula, that

$$R_n(y) = \sum_{t=1}^{n} I(0 < x_t \leq yn^{-1/2}) \overset{d}{\approx} n\int_0^1 I(0 < W(r) \leq yn^{-1})dr$$

$$= n\int_{-\infty}^{\infty} I(0 < s \leq yn^{-1})L(1,s)ds \overset{d}{\approx} yL(1,0) \overset{d}{=} y|Z|, \tag{12}$$

where $\overset{d}{\approx}$ denotes "having a roughly similar distribution as $n \to \infty$", $L(t,s)$ denotes Brownian local time, and $Z \sim N(0,1)$. Note that the equivalence $L(1,0) \overset{d}{=} |Z|$ follows from Akonom (1993, p. 58). If that were the case, then the second moment of $R(y)$ would have to be $E(yZ)^2 = y^2$. However, by Equation (8), we find

$$\mu_2 = \mu_1 + y^2 = y\sqrt{2/\pi} + y^2, \tag{13}$$

and therefore this conjecture is false and $R(y)$ is not distributed as $y|Z|$.

Using the previous lemma, convergence results for $Y_{n1} = n^{1/2}\min_{\{t:1\leq t\leq n, x_t>0\}} x_t$ and $Z_{n1} = n^{1/2}\min_{1\leq t\leq n}|x_t|$ now can be proven:

Theorem 1. *Under Assumption 1,*

1. $\lim_{n\to\infty} P(Y_{n1} \le y)$ *is well-defined for all* $y \in \mathbb{R}$*, and the limit is Lipschitz continuous for* $y \in \mathbb{R}$*. Furthermore,* Y_{n1}^{-1} *converges in distribution.*

2. $\lim_{n\to\infty} P(Z_{n1} \le y)$ *is well-defined for all* $y \in \mathbb{R}$*, and the limit is Lipschitz continuous for* $y \in \mathbb{R}$*. Furthermore,* Z_{n1}^{-1} *converges in distribution.*

Since the limit results of Theorem 1 are based on the observation of Equations (5) and (7), the limits found in Theorem 1 do not depend on the distribution of Δx_t, because the distributions of $R(y)$ and $S(y)$ do not depend on the distribution of Δx_t.

Theorem 1 does not rule out the possibility that some of the probability mass of Y_{n1} or Z_{n1} escapes to infinity asymptotically. Therefore, the limit measures of Y_{n1} and Z_{n1} are not necessarily probability measures. This is the reason that Theorem 1 is not formulated as a convergence in distribution result. However, Theorem 1 implies that any continuous and bounded function of Y_{n1} or Z_{n1} converges in distribution.

Chung (2001, p. 85) uses the term "vague convergence" for the concept of convergence of a sequence of probability measures to a limit measure that is not necessarily a probability measure. Therefore, the result of Theorem 1 implies that Y_{n1} and Z_{n1} converge vaguely in Chung's sense.

Note that not all probability mass of Y_{n1} and Z_{n1} escapes to infinity. After all, if all probability mass of Y_{n1} escaped to infinity, we would have $\lim_{n\to\infty} P(Y_{n1} \le y) = 0$ for all $y > 0$, implying that for all $y > 0$, by Equation (7) and Lemma 1,

$$0 = \lim_{n\to\infty} P(Y_{n1} \le y) = P(R(y) > 1/2) = 0, \tag{14}$$

and because $R(y) \in \mathbb{N}$, this would imply that $R(y) = 0$ a.s., which contradicts our earlier finding that $\mu_1 > 0$ for $y > 0$. The same argument holds for Z_{n1}.

4 Joint convergence of occupation times for small intervals

In this section we derive multivariate equivalents to the results from the previous section. Consider $(Y_{n1}, Y_{n2})'$, where Y_{n1} is as before and Y_{n2} is $n^{1/2}$ times the 2nd smallest positive value for x_t. Assume that n is large enough for $\{x_t : x_t > 0, t = 1, \ldots, n\}$ to have at least 2 elements, so that $(Y_{n1}, Y_{n2})'$ are well-defined. The joint distribution of $(Y_{n1}, Y_{n2})'$ then satisfies, for all $y_1, y_2 \in \mathbb{R}$,

$$P(Y_{n1} \leq y_1, Y_{n2} \leq y_2) = P(R_n(y_1) > 1/2, R_n(y_2) > 3/2). \tag{15}$$

Analogously, assuming that n is large enough for $\{x_t : x_t > 0, t = 1, \ldots, n\}$ to have at least m elements, defining Y_{ni} as the ith smallest positive value for $n^{1/2}x_t$ for $i = 1, \ldots, m$, for all $y_1, y_2, \ldots, y_m \in \mathbb{R}$

$$P(Y_{n1} \leq y_1, \ldots, Y_{nm} \leq y_m) = P(R_n(y_i) > i - 1/2 \ \ \forall i \in \{1, \ldots, m\}). \tag{16}$$

A similar observation can be made for Z_{ni}, which is defined as the ith smallest value of $n^{1/2}|x_t|$. With these definitions and observations in place, we can now find the following joint convergence result for the occupation times of Y_{ni} and Z_{ni}:

Lemma 2. *Under Assumption 1,*

1. *for all $(y_1, y_2, \ldots, y_m)' \in \mathbb{R}^m$ there exists a random variable $(R(y_1), R(y_2), \ldots, R(y_m))'$ such that*

$$(R_n(y_1), R_n(y_2), \ldots, R_n(y_m))' \xrightarrow{d} (R(y_1), R(y_2), \ldots, R(y_m))'; \tag{17}$$

2. *for all $(y_1, y_2, \ldots, y_m)' \in \mathbb{R}^m$ there exists a random variable $(S(y_1), S(y_2), \ldots, S(y_m))'$ such that*

$$(S_n(y_1), S_n(y_2), \ldots, S_n(y_m))' \xrightarrow{d} (S(y_1), S(y_2), \ldots, S(y_m))'. \tag{18}$$

Inspecting the proof of Lemma 2 again reveals that the distributions of $(R(y_1), R(y_2), \ldots, R(y_m))'$ and $(S(y_1), S(y_2), \ldots, S(y_m))'$ do not depend on the distribution of Δx_t.

Our multivariate results for $(Y_{n1}, \ldots, Y_{nm})'$ and $(Z_{n1}, \ldots, Z_{nm})'$ are as follows:

Theorem 2. *Under Assumption 1,*

1. $\lim_{n\to\infty} P(Y_{n1} \leq y_1, Y_{n2} \leq y_2, \ldots, Y_{nm} \leq y_m)$ *is well-defined for all* $(y_1, \ldots, y_m)' \in \mathbb{R}^m$, *and the limit is Lipschitz continuous for* $(y_1, \ldots, y_m)' \in \mathbb{R}^m$. *Furthermore,* $(Y_{n1}^{-1}, Y_{n2}^{-1}, \ldots, Y_{nm}^{-1})'$ *converges in distribution.*

2. $\lim_{n\to\infty} P(Z_{n1} \leq y_1, Z_{n2} \leq y_2, \ldots, Z_{nm} \leq y_m)$ *is well-defined for all* $(y_1, \ldots, y_m)' \in \mathbb{R}^m$, *and the limit is Lipschitz continuous for* $(y_1, \ldots, y_m)' \in \mathbb{R}^m$. *Furthermore,* $(Z_{n1}^{-1}, Z_{n2}^{-1}, \ldots, Z_{nm}^{-1})'$ *converges in distribution.*

Similarly to Theorem 1, the limits found in Theorem 2 do not depend on the distribution of Δx_t because they are induced by the limit distributions $(R(y_1), R(y_2), \ldots, R(y_m))'$ and $(S(y_1), S(y_2), \ldots, S(y_m))'$.

5 Summations of negative powers

This section considers statistics

$$n^{-q/2} \sum_{t=1}^{n} x_t^{-q} I(x_t > 0) \tag{19}$$

and

$$n^{-q/2} \sum_{t=1}^{n} |x_t|^{-q} \tag{20}$$

for $q > 1$. These statistics have also been considered by Pötscher (2013); in this section we will find their limit distributions using Theorem 2 of the previous section. The idea here is that the statistic $n^{-q/2}\sum_{t=1}^{n} x_t^{-q} I(x_t > 0)$ can be written as $\sum_{t=1}^{M_n} Y_{nt}^{-q}$, where M_n is the number of positive x_t, and that the last statistic is asymptotically close to $\sum_{t=1}^{m} Y_{nt}^{-q}$ for large m. By the joint convergence in distribution result of Theorem 2, and letting $(Y_1^{-1}, \ldots, Y_m^{-1})'$ denote a random variable that has the limit distribution of $(Y_{n1}^{-1}, \ldots, Y_{nm}^{-1})'$, we find that $\sum_{t=1}^{m} Y_{nt}^{-q} \xrightarrow{d} \sum_{t=1}^{m} Y_t^{-q}$, and since m was arbitrary, we find the limit distribution as $\sum_{t=1}^{\infty} Y_t^{-q}$. A similar result holds for $n^{-q/2}\sum_{t=1}^{n} |x_t|^{-q}$, defining $(Z_1^{-1}, \ldots, Z_m^{-1})'$ analogously. This argument then leads to the following result:

Theorem 3. *For $q > 1$, under Assumption 1,*

$$n^{-q/2}\sum_{t=1}^{n} x_t^{-q} I(x_t > 0) \xrightarrow{d} \sum_{t=1}^{\infty} Y_t^{-q} \tag{21}$$

and

$$n^{-q/2}\sum_{t=1}^{n} |x_t|^{-q} \xrightarrow{d} \sum_{t=1}^{\infty} Z_t^{-q}, \tag{22}$$

and the limit distributions do not depend on the distribution of the innovations Δx_t.

6 Simulation results

We conducted a small simulation experiment to illustrate the main theorems. We simulated $\min_{\{t:1\leq t\leq n, x_t>0\}} x_t$, $\min_{1\leq t\leq n} |x_t|$, $n^{-1}\sum_{t=1}^{n} x_t^{-2} I(x_t > 0)$ and $n^{-1}\sum_{t=1}^{n} x_t^{-2}$ for various values on n and i.i.d. $N(0,1)$ distributed Δx_t, and reported the percentage points in Tables 1, 2, 3, and 4. For all four statistics, the convergence of the distribution was rapid. Note that the "NA" for $n = 100$ in Table 1 for the 95th percentage point corresponds to the occurrence in more than 5% of cases of an integrated process that was always negative.

Appendix: Simulation results

Table 1: Simulation results for $n^{1/2} \min_{\{t:1\le t\le n, x_t>0\}} x_t$.

n	No. of replications	5%	10%	25%	50%	75%	90%	95%
100	10^7	0.070	0.146	0.419	1.142	2.990	8.296	NA
1000	10^6	0.067	0.139	0.401	1.103	2.924	7.896	16.065
10,000	10^5	0.067	0.138	0.396	1.095	2.884	7.766	15.664
100,000	10^5	0.066	0.137	0.396	1.094	2.920	7.828	15.835

Table 2: Simulation results for $n^{1/2} \min_{1\le t\le n} |x_t|$.

n	No. of replications	5%	10%	25%	50%	75%	90%	95%
100	10^7	0.035	0.073	0.209	0.571	1.489	3.888	7.122
1000	10^6	0.033	0.070	0.201	0.553	1.460	3.943	7.919
10,000	10^5	0.034	0.070	0.199	0.550	1.460	3.917	7.854
100,000	10^5	0.033	0.069	0.199	0.550	1.462	3.962	7.922

Table 3: Simulation results for $n^{-1}\sum_{t=1}^{n} x_t^{-2} I(x_t > 0)$.

n	No. of replications	5%	10%	25%	50%	75%	90%	95%
100	10^7	0.000	0.045	0.256	1.399	7.958	54.230	219.790
1000	10^6	0.012	0.041	0.267	1.517	8.767	59.559	242.223
10,000	10^5	0.010	0.041	0.271	1.553	9.019	61.381	243.012
100,000	10^5	0.010	0.040	0.265	1.559	9.001	61.915	254.565

Table 4: Simulation results for $n^{-1}\sum_{t=1}^{n} x_t^{-2}$.

n	No. of replications	5%	10%	25%	50%	75%	90%	95%
100	10^7	0.072	0.184	1.029	5.632	31.939	217.331	877.243
1000	10^6	0.044	0.162	1.066	6.073	34.930	239.072	968.151
10,000	10^5	0.042	0.160	1.071	6.155	35.852	237.745	939.627
100,000	10^5	0.039	0.155	1.068	6.218	35.930	244.387	995.949

Appendix: Mathematical proofs

In this appendix, for brevity we will set $I_{t1} = I(0 < x_t \leq yn^{-1/2})$ and $I_{t2} = I(|x_t| \leq yn^{-1/2})$. We will also define $I_{t,y,1} = I(0 < x_t \leq yn^{-1/2})$ and $I_{t,y,2} = I(|x_t| \leq yn^{-1/2})$ whenever three instead of two arguments are used for I.

Proof of Lemma 1

The proof of Lemma 1 relies on Lemmas 3 to 10 below.

Lemma 3. *A random sequence $X_n \in \mathbb{R}$ converges in distribution to a random variable X with moments ζ_p if (1) EX_n^p converges to a limit ζ_p for all $p \in \mathbb{N}$; and (2) $\sum_{p=1}^{\infty} \zeta_{2p}^{-1/(2p)} = \infty$.*

Proof of Lemma 3: This result follows from Fréchet and Shohat (1931) as quoted on page 2 of Lin (2017), together with Lin's Theorem 1 and condition (h7). □

Lemma 4. *As $n \to \infty$, for $p \geq 2$,*

$$n^{-p/2}\sum_{t_1=1}^{n}\dots\sum_{t_p=1}^{n} I(t_1 < t_2)\dots I(t_{p-1} < t_p)t_1^{-1/2}(t_2 - t_1)^{-1/2}\dots(t_p - t_{p-1})^{-1/2}$$

$$\to \int_0^1 \int_0^{s_p} \dots \int_0^{s_2} s_1^{-1/2}(s_2 - s_1)^{-1/2} \dots (s_{p-1} - s_{p-2})^{-1/2}(s_p - s_{p-1})^{-1/2} ds_1 \dots ds_p.$$

Proof of Lemma 4: Note that, by letting [.] denote the floor function and setting $t_1 = j_1$ and $t_i - t_{i-1} = j_i$ for $i = 2, \dots, p$, and then $j_i = nx_i + 1$ for $i = 1, \dots, p$,

$$n^{-p/2} \sum_{t_1=1}^{n} \dots \sum_{t_p=1}^{n} I(t_1 < t_2) \dots I(t_{p-1} < t_p) t_1^{-1/2}(t_2 - t_1)^{-1/2} \dots (t_p - t_{p-1})^{-1/2}$$

$$= n^{-p/2} \sum_{j_1=1}^{n} \dots \sum_{j_p=1}^{n} I(1 \leq j_1 + j_2 \leq n) \dots I(1 \leq j_1 + \dots + j_p \leq n) j_1^{-1/2} j_2^{-1/2} \dots j_p^{-1/2}$$

$$= n^{-p/2} \int_{j_1=1}^{n+1} \dots \int_{j_p=1}^{n+1} I(1 \leq [j_1]+[j_2] \leq n) \dots I(1 \leq [j_1]+\dots+[j_p] \leq n)[j_1]^{-1/2}[j_2]^{-1/2} \dots [j_p]^{-1/2} dj_1 \dots dj_p$$

$$= n^{p/2} \int_{x_1=0}^{1} \dots \int_{x_p=0}^{1} I(1 \leq [nx_1+1]+[nx_2+1] \leq n) \dots I(1 \leq [nx_1+1]+\dots+[nx_p+1] \leq n)$$

$$\times [nx_1 + 1]^{-1/2}[nx_2 + 1]^{-1/2} \dots [nx_p + 1]^{-1/2} dx_1 \dots dx_p. \quad (23)$$

Pointwise for $(x_1, \dots, x_p) \in (0, 1]^p$,

$$n^{p/2} I(1 \leq [nx_1+1]+[nx_2+1] \leq n) \dots I(1 \leq [nx_1+1]+\dots+[nx_p+1] \leq n)[nx_1+1]^{-1/2}[nx_2+1]^{-1/2} \dots [nx_p+1]^{-1/2}$$

$$\to I(x_1 + x_2 \leq 1) \dots I(x_1 + ... + x_p \leq 1) x_1^{-1/2} x_2^{-1/2} \dots x_p^{-1/2}$$

and therefore, by the dominated convergence theorem, it suffices to find an integrable dominating function. To find the dominating function, note that because $[x + 1] \geq x$

$$n^{p/2} I(1 \leq [nx_1+1]+[nx_2+1] \leq n) \dots I(1 \leq [nx_1+1]+\dots+[nx_p+1] \leq n)[nx_1+1]^{-1/2}[nx_2+1]^{-1/2} \dots [nx_p+1]^{-1/2}$$

$$\leq x_1^{-1/2} x_2^{-1/2} \dots x_p^{-1/2},$$

which is integrable. Therefore, the limit of the statistic of Equation (23) is

$$\int_0^1 \dots \int_0^1 I(x_1 + x_2 \leq 1) \dots I(x_1 + ... + x_p \leq 1) x_1^{-1/2} x_2^{-1/2} \dots x_p^{-1/2} dx_1 \dots dx_p$$

$$= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} I(0 \le x_1 \le 1) \dots I(0 \le x_p \le 1) I(x_1 + x_2 \le 1) \dots I(x_1 + ... + x_p \le 1)$$
$$\times\, x_1^{-1/2} x_2^{-1/2} \dots x_p^{-1/2} dx_1 \dots dx_p.$$

Now set $x_1 = s_1$, $x_1 + x_2 = s_2$, $x_1 + x_2 + x_3 = s_3$, etc.. Then the last expression can be rewritten as

$$\int_0^1 \dots \int_0^1 I(0 \le s_1 \le 1) I(s_1 \le s_2 \le 1) ... I(s_{p-1} \le s_p \le 1)$$
$$\times s_1^{-1/2} (s_2 - s_1)^{-1/2} \dots (s_p - s_{p-1})^{-1/2} ds_1 \dots ds_p$$
$$= \int_0^1 \int_0^{s_p} \dots \int_0^{s_2} s_1^{-1/2} (s_2 - s_1)^{-1/2} \dots (s_{p-1} - s_{p-2})^{-1/2} (s_p - s_{p-1})^{-1/2} ds_1 \dots ds_p,$$

which is the result as stated in the lemma. □

Lemma 5. *For* $p \ge 1$, *setting* $s_0 = 0$,

$$\int_{s_p=0}^{s_p=1} \int_{s_{p-1}=0}^{s_p} \dots \int_{s_1=0}^{s_1=s_2} (s_p - s_{p-1})^{-1/2} (s_{p-1} - s_{p-2})^{-1/2} \dots (s_2 - s_1)^{-1/2} s_1^{-1/2} ds_1 \dots ds_p$$
$$= (\Gamma(1/2))^p / \Gamma(p/2 + 1).$$

Proof of Lemma 5: Equations (2.2) and (2.3) on p. 44 and 45 of Miller and Ross (1993) state the following two properties of the Riemann-Liouville fractional integral:

(R1) For $\alpha > 0$, $D_t^{-\alpha} f(t) = (\Gamma(\alpha))^{-1} \int_0^t (t - \tau)^{\alpha - 1} f(\tau) d\tau$.

(R2) For $\alpha > 0$ and $\beta > -1$, $D_t^{-\alpha} t^\beta = \Gamma(\beta + 1) t^{\alpha + \beta} / \Gamma(\alpha + \beta + 1)$.

For $p = 1$, we use (R1) and (R2) using $\alpha = 1$, $\beta = -1/2$, and $f(\tau) = \tau^{-1/2}$, find that

$$\int_0^1 s_1^{-1/2} ds_1 = \Gamma(1) D_t^{-1} t^{-1/2}|_{t=1} = \Gamma(1/2)/\Gamma(3/2) = 2.$$

For $p = 2$, we have

$$\int_0^1 \int_0^{s_2} (s_2 - s_1)^{-1/2} s_1^{-1/2} ds_1 ds_2 = \int_0^1 \Gamma(1/2) D_{s_2}^{-1/2} s_2^{-1/2} ds_2 = \Gamma(1/2) D_t^{-1} D_t^{-1/2} t^{-1/2}|_{t=1},$$

by applying (R1) twice (with $\alpha = 1/2$, $t = s_2$, $\tau = s_1$, $f(\tau) = \tau^{-1/2}$ and with $\alpha = 1$, $\tau = s_2$, $f(\tau) = D_\tau^{-1/2}\tau^{-1/2}$). Setting $\alpha = 3/2$ and $\beta = -1/2$, we apply (R2) to the above expression and find that it equals

$$\Gamma(1/2)\Gamma(1/2)t/\Gamma(2)|_{t=1} = (\Gamma(1/2))^2/\Gamma(2) = \pi.$$

For $p = 3$, note that

$$\int_0^1 \int_0^{s_3} (s_3 - s_2)^{-1/2} \int_0^{s_2} (s_2 - s_1)^{-1/2} s_1^{-1/2} ds_1 ds_2 ds_3$$

$$= \int_0^1 \int_0^{s_3} (s_3 - s_2)^{-1/2} \Gamma(1/2) D_{s_2}^{-1/2} s_2^{-1/2} ds_2 ds_3.$$

Applying (R1) twice (with $\alpha = 1/2$, $t = s_3$, $\tau = s_2$, $f(\tau) = D_\tau^{-1/2}\tau^{-1/2}$ and with $\alpha = 1$, $\tau = s_3$, $f(\tau) = D_\tau^{-1}\tau^{-1/2}$) gives

$$\int_0^1 \int_0^{s_3} (s_3 - s_2)^{-1/2} \Gamma(1/2) D_{s_2}^{-1/2} s_2^{-1/2} ds_2 ds_3 = \int_0^1 (\Gamma(1/2))^2 D_{s_3}^{-1} s_3^{-1/2} ds_3 = (\Gamma(1/2))^2 D_t^{-2} t^{-1/2}|_{t=1}.$$

By setting $\alpha = 2$ and $\beta = -1/2$, (R2) implies that the expression is equal to

$$(\Gamma(1/2))^2\Gamma(1/2)t^{3/2}/\Gamma(5/2)|_{t=1} = (\Gamma(1/2))^3/\Gamma(5/2) = 4\pi/3.$$

Reasoning accordingly, we now find that the p-fold integral is

$$\int_0^1 (\Gamma(1/2))^{p-1} D_{s_p}^{-p/2+1/2} s_p^{-1/2} ds_p = D_t^{-1}(\Gamma(1/2))^{p-1} D_t^{-p/2+1/2} t^{-1/2}|_{t=1},$$

by (R1). By setting $\alpha = p/2 + 1/2$ and $\beta = -1/2$, (R2) implies that the above expression is equal to

$$(\Gamma(1/2))^{p-1}\Gamma(1/2)t^{p/2}/\Gamma(p/2+1)|_{t=1} = (\Gamma(1/2))^p/\Gamma(p/2+1),$$

thereby completing the proof. □

Lemma 6. *Assume Assumption 1 holds. Then we have*

$$\lim_{n\to\infty} E(\sum_{t=1}^{n} I_{t1}) = y\sqrt{2/\pi}$$

and

$$\lim_{n\to\infty} E(\sum_{t=1}^{n} I_{t2}) = 2y\sqrt{2/\pi}$$

and for $p \geq 2$,

$$\lim_{n\to\infty} \sum_{t_1=1}^{n} \dots \sum_{t_p=1}^{n} E(I_{t_1,1} \dots I_{t_p,1}) I(t_1 \neq t_2) I(t_2 \neq t_3) \dots I(t_{p-1} \neq t_p) = p! y^p 2^{-p/2} / \Gamma(p/2+1)$$

and

$$\lim_{n\to\infty} \sum_{t_1=1}^{n} \dots \sum_{t_p=1}^{n} E(I_{t_1,2} \dots I_{t_p,2}) I(t_1 \neq t_2) I(t_2 \neq t_3) \dots I(t_{p-1} \neq t_p) = p! y^p 2^{p/2} / \Gamma(p/2+1).$$

Proof of Lemma 6: We will first show the first assertion of the lemma. By the Taylor expansion, for some intermediate value $\xi_{tny} \in [0, yt^{-1/2}n^{-1/2}]$,

$$\mu_1 = \lim_{n\to\infty} E(\sum_{t=1}^{n} I_{t1}) = \lim_{n\to\infty} \sum_{t=1}^{n} (F_t(yt^{-1/2}n^{-1/2}) - F_t(0)) = \lim_{n\to\infty} yn^{-1/2} \sum_{t=1}^{n} t^{-1/2} f_t(\xi_{tny})$$
$$= 2y\phi(0) = y\sqrt{2/\pi}$$

because $n^{-1/2} \sum_{t=1}^{n} t^{-1/2} \to 2$ and because $\sup_{x\in\mathbb{R}} |f_t'(x)| < \infty$ and $\sup_{t\geq 1, x\in\mathbb{R}} |f_t(x) - \phi(x)| \to 0$ by assumption. The result for $\lim_{n\to\infty} E(\sum_{t=1}^{n} I_{t2})$ follows analogously.

To show the second assertion of the lemma, note that since there are $p!$ possible orderings of $\{t_1, t_2, \dots, t_p\}$,

$$\sum_{t_1=1}^{n} \dots \sum_{t_p=1}^{n} E(I_{t_1,1} \dots I_{t_p,1}) I(t_1 \neq t_2) I(t_2 \neq t_3) \dots I(t_{p-1} \neq t_p)$$

$$= p! \sum_{t_1=1}^{n} \dots \sum_{t_p=1}^{n} E(I_{t_1,1} \dots I_{t_p,1}) I(t_1 < t_2) I(t_2 < t_3) \dots I(t_{p-1} < t_p).$$

Letting $g_t(\cdot)$ denote the density of x_t, we have

$$E(I_{t_1,1} \dots I_{t_p,1}) = E(I(0 \leq x_{t_1} \leq yn^{-1/2}) \dots I(0 \leq x_{t_p} \leq yn^{-1/2}))$$

$$= E(I(0 \leq x_{t_p} - x_{t_{p-1}} + \dots + x_{t_2} - x_{t_1} + x_{t_1} \leq yn^{-1/2}) \dots I(0 \leq x_{t_1} \leq yn^{-1/2}))$$

$$= \int \dots \int I(0 \leq z_1 + z_2 + \dots + z_p \leq yn^{-1/2}) \dots I(0 \leq z_1 \leq yn^{-1/2})$$

$$\times g_{t_1}(z_1) g_{t_2-t_1}(z_2) \dots g_{t_p-t_{p-1}}(z_p) dz_1 \dots dz_p.$$

Since $g_t(z) = t^{-1/2} f_t(t^{-1/2} z)$, $\sup_{|z| \leq yn^{-1/2}} |g_t(z) - t^{-1/2}\phi(0)| = o(t^{-1/2})$ because

$$c_t = \sup_{|z| \leq yn^{-1/2}} |g_t(z) - t^{-1/2}\phi(0)| = \sup_{|z| \leq yn^{-1/2}} |t^{-1/2} f_t(t^{-1/2} z) - t^{-1/2}\phi(0)|$$

$$\leq t^{-1/2} \sup_{|z| \leq yn^{-1/2}} (|f_t(t^{-1/2} z) - f_t(0)| + |f_t(0) - \phi(0)|) = o(t^{-1/2})$$

because by assumption $\sup_{x \in \mathbb{R}} |f_t(x) - \phi(x)| \to 0$ and $\sup_{t \geq 1, x \in \mathbb{R}} |f_t'(x)| < \infty$. Therefore, approximating $g_{t_1}(z)$ by $t^{-1/2}\phi(0)$ gives

$$| \sum_{t_1=1}^{n} \dots \sum_{t_p=1}^{n} I(t_1 < t_2) \dots I(t_{p-1} < t_p) E(I_{t_1,1} \dots I_{t_p,1})$$

$$- \sum_{t_1=1}^{n} \dots \sum_{t_p=1}^{n} I(t_1 < t_2) \dots I(t_{p-1} < t_p) \int \dots \int I(0 \leq z_1 + z_2 + \dots + z_p \leq yn^{-1/2}) \dots I(0 \leq z_1 \leq yn^{-1/2})$$

$$\times t_1^{-1/2} \phi(0) g_{t_2-t_1}(z_2) \dots g_{t_p-t_{p-1}}(z_p) dz_1 \dots dz_p |$$

$$\leq \sum_{t_1=1}^{n} c_{t_1} \sum_{t_2=1}^{n} \dots \sum_{t_p=1}^{n} I(t_1 < t_2) \dots I(t_{p-1} < t_p) \int \dots \int I(0 \leq z_1 + z_2 + \dots + z_p \leq yn^{-1/2}) \dots I(0 \leq z_1 \leq yn^{-1/2})$$

$$\times g_{t_2-t_1}(z_2) \dots g_{t_p-t_{p-1}}(z_p) dz_1 \dots dz_p$$

$$= \sum_{t_1=1}^{n} c_{t_1} \sum_{t_2=1}^{n} \dots \sum_{t_p=1}^{n} I(t_1 < t_2) \dots I(t_{p-1} < t_p) (t_2 - t_1)^{-1/2} \dots (t_p - t_{p-1})^{-1/2}$$

$$\times \int \dots \int I(0 \le z_1 + z_2 + \dots + z_p \le yn^{-1/2}) \dots I(0 \le z_1 \le yn^{-1/2}) f_{t_2-t_1}(z_2) \dots f_{t_p-t_{p-1}}(z_p) dz_1 \dots dz_p$$

$$\le \sum_{t_1=1}^{n} c_{t_1} \sum_{t_2=1}^{n} (t_2 - t_1)^{-1/2} \dots \sum_{t_p=1}^{n} (t_p - t_{p-1})^{-1/2} I(t_1 < t_2) \dots I(t_{p-1} < t_p) (yn^{-1/2})^p (\sup_{t \ge 1, x \in \mathbb{R}} f_t(x))^{p-1}$$

$$= O(n^{-1/2} \sum_{t=1}^{n} c_t) = o(1).$$

Similarly, we can also approximate $g_{t_2-t_1}(z)$ by $(t_2-t_1)^{-1/2}\phi(0)$, $g_{t_3-t_2}(z)$ by $(t_3-t_2)^{-1/2}\phi(0)$, etc. and therefore

$$\sum_{t_1=1}^{n} \dots \sum_{t_p=1}^{n} I(t_1 < t_2) \dots I(t_{p-1} < t_p) E(I_{t_1,1} \dots I_{t_p,1})$$

$$= \sum_{t_1=1}^{n} \dots \sum_{t_p=1}^{n} I(t_1 < t_2) \dots I(t_{p-1} < t_p) \int \dots \int I(0 \le z_1 + z_2 + \dots + z_p \le yn^{-1/2}) \dots I(0 \le z_1 \le yn^{-1/2})$$

$$\times t_1^{-1/2} \phi(0)^p (t_2 - t_1)^{-1/2} \dots (t_p - t_{p-1})^{-1/2} dz_1 \dots dz_p + o(1).$$

Because

$$\int \dots \int I(0 \le z_1 + z_2 + \dots + z_p \le yn^{-1/2}) \dots I(0 \le z_1 \le yn^{-1/2}) dz_1 \dots dz_p = y^p n^{-p/2}, \quad (24)$$

it now follows that

$$\sum_{t_1=1}^{n} \dots \sum_{t_p=1}^{n} I(t_1 < t_2) \dots I(t_{p-1} < t_p) E(I_{t_1,1} \dots I_{t_p,1})$$

$$= o(1) + (yn^{-1/2}\phi(0))^p \sum_{t_1=1}^{n} \dots \sum_{t_p=1}^{n} I(t_1 < t_2) \dots I(t_{p-1} < t_p) t_1^{-1/2} (t_2 - t_1)^{-1/2} \dots (t_p - t_{p-1})^{-1/2},$$

and by Lemma 4, the last expression equals

$$o(1) + y^p \phi(0)^p \int_0^1 (s_p - s_{p-1})^{-1/2} \int_0^{s_{p-1}} (s_{p-1} - s_{p-2})^{-1/2} \dots \int_0^{s_2} s_1^{-1/2} (s_2 - s_1)^{-1/2} ds_1 \dots ds_p$$

and by Lemma 5, it now follows that

$$\lim_{n\to\infty}\sum_{t_1=1}^{n}\dots\sum_{t_p=1}^{n}E(I_{t_1,1}\dots I_{t_p,1})I(t_1\neq t_2)I(t_2\neq t_3)\dots I(t_{p-1}\neq t_p)$$

$$= p!\ y^p\phi(0)^p\ (\Gamma(1/2))^p/\Gamma(p/2+1)$$

$$= p!y^p 2^{-p/2}/\Gamma(p/2+1).$$

The result for I_{t2} follows analogously, except that in the result of Equation (24), $y^p n^{-p/2}$ needs to be replaced by $2^p y^p n^{-p/2}$. □

Lemma 7. *Assume Assumption 1 holds. For $b=1,2$, $p\geq 2$ and $j=1,2,\dots,p-1$, define*

$$h_{bnpj}=\sum_{t_1=1}^{n}\sum_{t_2=1}^{n}\cdots\sum_{t_p=1}^{n}E(I_{t_1,b}I_{t_2,b}\cdots I_{t_p,b})I(t_1\neq t_2)I(t_2\neq t_3)\cdots I(t_j\neq t_{j+1})$$

and for $b=1,2$ and $p\geq 1$, define $h_{bnp0}=E(\sum_{t=1}^{n}I_{tb})^p$. Then $H_{bpj}=\lim_{n\to\infty}h_{bnpj}$ is well-defined for $b=1,2$, $p\geq 1$ and $j=0,1,2,\dots,p-1$, and as a consequence, $\mu_p=H_{1p0}=\lim_{n\to\infty}h_{1np0}$ and $\nu_p=H_{2p0}=\lim_{n\to\infty}h_{2np0}$ are well-defined for $p\geq 1$. Also, for $b=1,2$, $p\geq 2$ and $j=0,\dots,p-2$,

$$H_{bpj}=H_{b,p-1,j}+H_{b,p,j+1}. \tag{25}$$

Proof of Lemma 7: For $b=1,2$, $p\geq 3$ and $j=1,2,\dots,p-2$, we can write

$$h_{bnpj}=\sum_{t_1=1}^{n}\sum_{t_2=1}^{n}\cdots\sum_{t_p=1}^{n}E(I_{t_1,b}I_{t_2,b}\cdots I_{t_p,b})I(t_1\neq t_2)\cdots I(t_j\neq t_{j+1})$$

$$=\sum_{t_1=1}^{n}\sum_{t_2=1}^{n}\cdots\sum_{t_p=1}^{n}E(I_{t_1,b}I_{t_2,b}\cdots I_{t_p,b})I(t_1\neq t_2)\cdots I(t_j\neq t_{j+1})(I(t_{j+1}=t_{j+2})+I(t_{j+1}\neq t_{j+2}))$$

$$=\sum_{t_1=1}^{n}\sum_{t_2=1}^{n}\cdots\sum_{t_{p-1}=1}^{n}E(I_{t_1,b}I_{t_2,b}\cdots I_{t_{p-1},b})I(t_1\neq t_2)\cdots I(t_j\neq t_{j+1})$$

$$+\sum_{t_1=1}^{n}\sum_{t_2=1}^{n}\cdots\sum_{t_p=1}^{n}E(I_{t_1,b}I_{t_2,b}\cdots I_{t_p})I(t_1\neq t_2)\cdots I(t_j\neq t_{j+1})I(t_{j+1}\neq t_{j+2})$$

$$= h_{b,n,p-1,j}+h_{b,n,p,j+1}, \tag{26}$$

where the third equality follows from the fact that $I_{t_{j+1},b}I_{t_{j+2},b} = I_{t_{j+1},b}$ when $t_{j+1} = t_{j+2}$, and we relabel the summation indices. This shows the result of Equation (26) for $b = 1, 2$, $p \geq 3$ and $j = 1, \ldots, p-2$. For $p \geq 2$ and $j = 0$, it is easy to see that we also have $h_{bnp0} = h_{b,n,p-1,0} + h_{bnp1}$ because

$$E(\sum_{t=1}^{n} I_{tb})^p = E(\sum_{t=1}^{n} I_{tb})^{p-1} + \sum_{t_1=1}^{n}\sum_{t_2=1}^{n}\cdots\sum_{t_p=1}^{n}E(I_{t_1,b}I_{t_2,b}\cdots I_{t_p,b})I(t_1\neq t_2).$$

Therefore, we conclude that the result of Equation (26) holds for $b = 1, 2$, $p \geq 2$, and $j = 0, \ldots, p-2$.

It follows from Lemma 6 that for $b = 1, 2$, $H_{b,p,p-1} = \lim_{n\to\infty} h_{b,n,p,p-1}$ exists for $p \geq 1$. To show that $H_{bpj} = \lim_{n\to\infty} h_{bnpj}$ exists for $b = 1, 2$, $p \geq 2$ and $j = 0, \ldots, p-2$, note that by setting $j = p-2$ in Equation (26), we now have for $b = 1, 2$, $p \geq 2$

$$h_{b,n,p,p-2} = h_{b,n,p-1,p-2} + h_{b,n,p,p-1}.$$

By taking limits, it now follows that $H_{b,p,p-2}$ exists because $H_{b,p,p-2} = H_{b,p-1,p-2} + H_{b,p,p-1}$. Repeating this argument for $j = p-3$, then $p-4$ etc. until $j = 0$ then shows that $\lim_{n\to\infty} h_{bnpj} = H_{bpj}$ exists for $b = 1, 2$, $p \geq 2$ and $j = 0, \ldots, p-2$. Therefore, the existence of H_{bpj} is now shown for $b = 1, 2$, $p \geq 1$, and $j = 0, \ldots, p-1$. The result of Equation (25) now follows by taking the limit as $n \to \infty$ in Equation (26). □

Lemma 8. *Assume Assumption 1 holds. Then for $p \geq 2$ and $0 \leq j \leq p-1$, $H_{1pj} = \Delta^j \mu_p$ and $H_{2pj} = \Delta^j \nu_p$.*

Proof of Lemma 8: Equation (25) states that $H_{bpj} = H_{b,p-1,j} + H_{b,p,j+1}$ for $b = 1, 2$, $p \geq 2$ and $j = 0, \ldots, p-2$. Setting $j + 1 = i$, we find $H_{bpi} = H_{b,p,i-1} - H_{b,p-1,i-1} = \Delta H_{b,p,i-1}$ for $b = 1, 2$, $p \geq 2$ and $i = 1, \ldots, p-1$. Repeating this equation gives, if $p \geq 2$ and $1 \leq i - 1 \leq p - 1$, $H_{bpi} = \Delta^2 H_{b,p,i-2}$. Therefore, repeating the equation k times, for $k \geq 0$ and $1 \leq i - k + 1 \leq p - 1$ we find $H_{bpi} = \Delta^k H_{b,p,i-k}$. Setting $k = i$ gives $H_{bpi} = \Delta^i H_{bp0}$ for $p \geq 2$. Therefore because $\mu_p = H_{1p0}$ and $\nu_p = H_{2p0}$, $H_{1pi} = \Delta^i \mu_p$ and $H_{2pi} = \Delta^i \nu_p$. This completes the proof. □

Lemma 9. *Assume Assumption 1 holds. Then for all* $p \geq 2$,

$$\Delta^{p-1}\mu_p = p! y^p 2^{-p/2} / \Gamma(p/2 + 1)$$

and

$$\Delta^{p-1}\nu_p = p! y^p 2^{p/2} / \Gamma(p/2 + 1).$$

Proof of Lemma 9: The result of Lemma 8 implies that $H_{1,p,p-1} = \Delta^{p-1}\mu_p$. Therefore, by Lemma 6 it now follows that

$$\Delta^{p-1}\mu_p = p! y^p 2^{-p/2} / \Gamma(p/2 + 1).$$

The result for ν_p is proven analogously. This completes the proof. □

Lemma 10. *Assume Assumption 1 holds. Then for the* μ_p *sequence as defined in Lemma 7, for* $p \geq 2$, $\mu_p \leq (2p)^p \max(1, y^p)$ *and* $\nu_p \leq 2^{2p} p^p \max(1, y^p)$.

Proof of Lemma 10: By Lemma 7, we have for $p \geq 2$ and $j = 0, \ldots, p-2$

$$H_{1pj} = H_{1,p-1,j} + H_{1,p,j+1} \leq 2 \max_{i_1=0,1} H_{1,p-1+i_1,j+i_1}$$

and applying this reasoning k times,

$$H_{1pj} \leq 4 \max_{i_1=0,1} \max_{i_2=0,1} H_{1,p-2+i_1+i_2,j+i_1+i_2} \leq \ldots \leq 2^k \max_{i_1=0,1} \ldots \max_{i_k=0,1} H_{1,p-k+i_1+\ldots+i_k,j+i_1+\ldots+i_k}.$$

Note that Lemma 7 states that $H_{1,p-k+i_1+\ldots+i_k,j+i_1+\ldots+i_k}$ is well-defined for $0 \leq j \leq p-k-1$. This is because H_{bpj} is well-defined for $b = 1, 2$, $p \geq 1$, and $j = 0, \ldots, p-1$ by Lemma 7, and $p - k + i_1 + \ldots + i_k \geq p - k \geq 1$ and $0 \leq j + i_1 + \ldots + i_k \leq j + k \leq p - 1$.

Therefore, setting $j = 0$ and $k = p - 1$,

$$\mu_p = H_{1p0} \leq 2^{p-1} \max_{i_1=0,1} \ldots \max_{i_{p-1}=0,1} H_{1,1+i_1+\ldots+i_{p-1},i_1+\ldots+i_{p-1}}$$

$$= 2^{p-1} \max_{0\leq i\leq p-1} H_{1,1+i,i} = 2^{p-1} \max_{1\leq i\leq p} H_{1,i,i-1}.$$

Because $H_{1,p,p-1} = p!2^{-p/2}y^p/\Gamma(p/2+1)$ by Lemma 6 , $p! \leq p^p$, $\Gamma(p/2+1) \geq 1$ and $2^{-p/2} \leq 1$ for $p \geq 2$,

$$2^{p-1} \max_{1\leq i\leq p} H_{i,i-1} \leq 2^{p-1} \max(1, y^p)p! \leq (2p)^p \max(1, y^p).$$

For ν_p the proof is analogous, except that the upper bound for ν_p of Lemma 6 is a factor 2^p larger. This completes the proof. □

With the above results in place, we can now complete the proof of Lemma 1:

Proof of Lemma 1: We first apply Lemma 3 to $X_n = \sum_{t=1}^n I_{t1}$ for part 1 of Lemma 1, and then to $X_n = \sum_{t=1}^n I_{t2}$ for part 2 of Lemma 1. Condition (1) of Lemma 3 holds because $EX_n^p = E(\sum_{t=1}^n I_{t1})^p$ converges to μ_p by Lemma 7. Condition (2) of Lemma 3 follows because $\sum_{p=1}^\infty \mu_{2p}^{-1/(2p)} = \infty$ because for $p \geq 2$, $\mu_{2p} \leq (4p)^{2p} \max(1, y^{2p})$ by Lemma 10. The value for μ_1 was calculated in Lemma 6, and the recursive relationship of Equation (8) was shown in Lemma 9. Therefore, the proof of the result for part 1 is now complete. The proof of part 2 of Lemma 1 is analogous. □

Proof of Theorem 1

Proof of Theorem 1: We will show part 1 of Theorem 1, and note that the proof for part 2 is analogous. Under Assumption 1, $R_n(y) = \sum_{t=1}^{n} I(0 \le x_t \le yn^{-1/2})$ converges in distribution to $R(y)$ by Lemma 1. Therefore, noting that

$$P(Y_{n1} \le y) = P(n^{1/2} \min_{\{t:1\le t\le n, x_t>0\}} x_t \le y) = 1 - P(\sum_{t=1}^{n} I(0 \le x_t \le yn^{-1/2}) \le 1/2)$$

and because $1/2$ is a continuity point of $R(y)$ for all $y \in \mathbb{R}$,

$$L(y) = \lim_{n\to\infty} P(n^{1/2} \min_{\{t:1\le t\le n, x_t>0\}} x_t \le y)$$

exists for all $y \in \mathbb{R}$. We will verify that $L(y)$ is Lipschitz continuous on $\mathbb{R}$. This follows because, for $y, y' \in \mathbb{R}$, $y \le y'$,

$$\begin{aligned}
&|L(y) - L(y')| \\
&= \lim_{n\to\infty} |P(n^{1/2} \min_{\{t:1\le t\le n, x_t>0\}} x_t \le y) - P(n^{1/2} \min_{\{t:1\le t\le n, x_t>0\}} x_t \le y')| \\
&= \lim_{n\to\infty} P(y < n^{1/2} \min_{\{t:1\le t\le n, x_t>0\}} x_t \le y') \\
&\le \limsup_{n\to\infty} P(\exists t \in \{1, \dots, n\} : y < n^{1/2} x_t \le y') \\
&= \limsup_{n\to\infty} P(\sum_{t=1}^{n} I(yn^{-1/2} \le x_t \le y'n^{-1/2}) > 1/2) \\
&\le 2 \limsup_{n\to\infty} \sum_{t=1}^{n} P(yn^{-1/2} \le x_t \le y'n^{-1/2}) \\
&= 2 \limsup_{n\to\infty} \sum_{t=1}^{n} (F_t(y'n^{-1/2}t^{-1/2}) - F_t(yn^{-1/2}t^{-1/2})) \\
&\le 2 \limsup_{n\to\infty} \sum_{t=1}^{n} |y - y'| n^{-1/2} t^{-1/2} \sup_{t\ge 1} \sup_{x\in\mathbb{R}} f_t(x)
\end{aligned}$$

$$\leq 2 \sup_{t\geq 1} \sup_{x\in\mathbb{R}} f_t(x)|y-y'| \sup_{n\geq 1} n^{-1/2} \sum_{t=1}^{n} t^{-1/2}, \tag{27}$$

where the first equality follows from the definition of $L(\cdot)$, the second inequality is the Markov inequality, and the third inequality follows from the mean value theorem.

To show that Y_{n1}^{-1} converges in distribution, note that for all $z_1 > 0$,

$$P(Y_{n1}^{-1} \leq z_1) = P(Y_{n1} \geq z_1^{-1}) = 1 - P(Y_{n1} \leq z_1^{-1})$$

converges as $n \to \infty$, and the limit is continuous at any $z_1 > 0$. Furthermore,

$$\lim_{z_1\to\infty} \lim_{n\to\infty} P(Y_{n1}^{-1} \leq z_1) = 1$$

because using the reasoning of Equation (27),

$$\lim_{z_1\to\infty} \limsup_{n\to\infty} |P(Y_{n1}^{-1} \leq z_1) - 1| = \lim_{z_1\to\infty} \limsup_{n\to\infty} |P(Y_{n1} \geq z_1^{-1}) - P(Y_{n1} \geq 0)|$$

$$\leq \lim_{z_1\to\infty} \limsup_{n\to\infty} P(0 \leq Y_{n1} \leq z_1^{-1}) \leq C \lim_{z_1\to\infty} z_1^{-1} = 0,$$

while for $z_1 < 0$, $P(Y_{n1}^{-1} \leq z_1) = 0$. This implies that $P(Y_{n1}^{-1} \leq z_1)$ converges to a well-defined limit distribution, and Y_{n1}^{-1} converges in distribution. The second part of the theorem is proven analogously. □

Proof of Lemma 2

Lemmas 11 to 17 are used for the proof of Lemma 2.

Lemma 11. *A random sequence $X_n \in \mathbb{R}^m$ converges in distribution to a random variable X if (1) for all $\lambda \neq 0$, $E(\lambda' X_n)^p$ converges to a limit $\zeta_{\lambda,p}$ for all $p \in \mathbb{N}$; and (2) $\sum_{p=1}^{\infty} \zeta_{\lambda,2p}^{-1/(2p)} = \infty$ for all $\lambda \neq 0$.*

Proof of Lemma 11: Since $\lambda' X_n$ satisfies the conditions of Lemma 3 for all $\lambda \in \mathbb{R}^m$, $\lambda \neq 0$, it follows that $\lambda' X_n \xrightarrow{d} Y_\lambda$ for some random variable Y_λ all $\lambda \in \mathbb{R}^m$, $\lambda \neq 0$. Therefore, $E\exp(i\lambda' X_n)$ converges pointwise in λ to a limit $\psi(\lambda)$, and we only need to show that $Y_\lambda \stackrel{d}{=} \lambda' X$ for some random variable X. It follows from Theorem 2.13 of Van der Vaart (2000, p. 14) that $\psi(\lambda)$ is the characteristic function of a random variable X if $\psi(\lambda)$ is continuous at 0. To show this, note that because $|\exp(ix) - \exp(iy)| \leq |x - y|$ for $x, y \in \mathbb{R}$,

$$|\psi(\lambda_1) - \psi(\lambda_2)| \leq \lim_{n\to\infty} |E\exp(i\lambda_1' X_n) - E\exp(i\lambda_2' X_n)|$$

$$\leq |\lambda_1 - \lambda_2| \lim_{n\to\infty} E|X_n| = |\lambda_1 - \lambda_2| \lim_{n\to\infty} (EX_n' X_n)^{1/2},$$

and, defining s_i as a vector of zeros except for a 1 at spot i,

$$\lim_{n\to\infty} EX_n' X_n = \lim_{n\to\infty} \sum_{i=1}^{m} E(s_i' X_n)^2$$

is well-defined because by assumption, $\lim_{n\to\infty} E(s_i' X_n)^2 = \zeta_{s_i,2}$ is well-defined. Therefore, the lemma is now proven. □

Lemma 12. *Assume Assumption 1 holds. Then for $p \geq 2$,*

$$\lim_{n\to\infty} \sum_{t_1=1}^{n} \cdots \sum_{t_p=1}^{n} E(I_{t_1,y_1,1} I_{t_2,y_2,1} \cdots I_{t_p,y_p,1}) I(t_1 < t_2) \cdots I(t_{p-1} < t_p)$$

$$= 2^{-p/2} \prod_{j=1}^{p} y_j / \Gamma(p/2 + 1)$$

and

$$\lim_{n\to\infty} \sum_{t_1=1}^{n} \cdots \sum_{t_p=1}^{n} E(I_{t_1,y_1,2} I_{t_2,y_2,2} \cdots I_{t_p,y_p,2}) I(t_1 < t_2) \cdots I(t_{p-1} < t_p)$$

$$= 2^{p/2} \prod_{j=1}^{p} y_j / \Gamma(p/2 + 1).$$

Proof of Lemma 12. We write that

$$
\begin{aligned}
& E(I_{t_1,y_1,1}I_{t_2,y_2,1}\cdots I_{t_p,y_p,1}) \\
& = E(I(0 \le x_{t_1} \le n^{-1/2}y_1)\cdots I(0 \le x_{t_p} \le n^{-1/2}y_p)) \\
& = \int\cdots\int I(0 \le z_1+z_2+\cdots+z_p \le n^{-1/2}y_p)\cdots I(0 \le z_1 \le n^{-1/2}y_1) \\
& \times g_{t_1}(z_1)g_{t_2-t_1}(z_2)\cdots g_{t_p-t_{p-1}}(z_p)dz_p\cdots dz_1.
\end{aligned}
$$

We follow a similar argument as in the proof of Lemma 6 and approximate $g_{t_1}(z)$ by $t_1^{-1/2}\phi(0)$ and $g_{t_j-t_{j-1}}(z)$ by $(t_j-t_{j-1})^{-1/2}\phi(0)$ for $j=2,3,\cdots,p$ and write that

$$
\begin{aligned}
& \lim_{n\to\infty}\sum_{t_1=1}^{n}\cdots\sum_{t_p=1}^{n} I(t_1<t_2)I(t_2<t_3)\cdots I(t_{p-1}<t_p)E(I_{t_1,y_1,1}\cdots I_{t_p,y_p,1}) \\
= & \lim_{n\to\infty}\sum_{t_1=1}^{n}\cdots\sum_{t_p=1}^{n} I(t_1<t_2)\cdots I(t_{p-1}<t_p)\int\cdots\int I(0\le z_1+\cdots+z_p\le n^{-1/2}y_p)\cdots I(0\le z_1\le n^{-1/2}y_1) \\
& \times\phi(0)^p t_1^{-1/2}(t_2-t_1)^{-1/2}\cdots(t_p-t_{p-1})^{-1/2}dz_p\cdots dz_1.
\end{aligned}
$$

Since

$$
\int\cdots\int I(0\le z_1+\cdots+z_p\le n^{-1/2}y_p)\cdots I(0\le z_1\le n^{-1/2}y_1)dz_p\cdots dz_1 = n^{-p/2}\prod_{j=1}^{p}y_j, \quad (28)
$$

we have

$$
\begin{aligned}
& \lim_{n\to\infty}\sum_{t_1=1}^{n}\cdots\sum_{t_p=1}^{n} I(t_1<t_2)I(t_2<t_3)\cdots I(t_{p-1}<t_p)E(I_{t_1,y_1,1}\cdots I_{t_p,y_p,1}) \\
= & \phi(0)^p\prod_{j=1}^{p}y_j\lim_{n\to\infty}n^{-p/2}\sum_{t_1=1}^{n}\cdots\sum_{t_p=1}^{n} I(t_1<t_2)\cdots I(t_{p-1}<t_p)t_1^{-1/2}(t_2-t_1)^{-1/2}\cdots(t_p-t_{p-1})^{-1/2}.
\end{aligned}
$$

Lemmas 4 and 5 now imply, by noting that $\phi(0)^p=(2\pi)^{-p/2}$ and $(\Gamma(1/2))^p=\pi^{p/2}$, that

$$
\lim_{n\to\infty}\sum_{t_1=1}^{n}\cdots\sum_{t_p=1}^{n} E(I_{t_1,y_1,1}I_{t_2,y_2,1}\cdots I_{t_p,y_p,1})I(t_1<t_2)\cdots I(t_{p-1}<t_p)
$$

$$= \phi(0)^p \prod_{j=1}^{p} y_j \int_0^1 \int_0^{s_p} \cdots \int_0^{s_2} s_1^{-1/2} (s_2 - s_1)^{-1/2} \cdots (s_p - s_{p-1})^{-1/2} ds_1 ds_2 \cdots ds_p$$

$$= 2^{-p/2} \prod_{j=1}^{p} y_j / \Gamma(p/2 + 1).$$

The proof of the second result of the lemma is analogous, except that the equivalent of Equation (28) now receives an additional 2^p factor. □

Lemma 13. *Assume Assumption 1 holds. Then we have*

$$\lim_{n\to\infty} E(\sum_{t_1=1}^{n} I_{t_1,y_1,1}) = y_1 \sqrt{2/\pi}$$

and

$$\lim_{n\to\infty} E(\sum_{t_1=1}^{n} I_{t_1,y_1,2}) = 2y_1 \sqrt{2/\pi}$$

and for $p \geq 2$,

$$\lim_{n\to\infty} \sum_{t_1=1}^{n} \sum_{t_2=1}^{n} \cdots \sum_{t_p=1}^{n} E(I_{t_1,y_1,1} I_{t_2,y_2,1} \cdots I_{t_p,y_p,1}) I(t_1 \neq t_2) I(t_2 \neq t_3) \cdots I(t_{p-1} \neq t_p)$$

$$= p! 2^{-p/2} \prod_{j=1}^{p} y_j / \Gamma(p/2 + 1)$$

and

$$\lim_{n\to\infty} \sum_{t_1=1}^{n} \sum_{t_2=1}^{n} \cdots \sum_{t_p=1}^{n} E(I_{t_1,y_1,2} I_{t_2,y_2,2} \cdots I_{t_p,y_p,2}) I(t_1 \neq t_2) I(t_2 \neq t_3) \cdots I(t_{p-1} \neq t_p)$$

$$= p! 2^{p/2} \prod_{j=1}^{p} y_j / \Gamma(p/2 + 1).$$

Proof of Lemma 13. The first two assertions of the lemma follows from Lemma 6 when we set $y = y_1$. The third result follows immediately from noting that

$$\lim_{n\to\infty}\sum_{t_1=1}^{n}\sum_{t_2=1}^{n}\cdots\sum_{t_p=1}^{n} E(I_{t_1,y_1,1}I_{t_2,y_2,1}\cdots I_{t_p,y_p,1})I(t_1\neq t_2)I(t_2\neq t_3)\cdots I(t_{p-1}\neq t_p)$$

$$= p!\ 2^{-p/2}\prod_{j=1}^{p} y_j/\Gamma(p/2+1),$$

because there are $p!$ orderings of $\{t_1, t_2, \ldots, t_p\}$, and Lemma 12 ensures that any possible ordering of $\{y_1, y_2, \ldots, y_p\}$ gives the same limit result. The fourth result follows analogously.

□

Lemma 14. *Assume Assumption 1 holds. For $b = 1, 2$, $p \geq 2$ and $j = 1, 2, \ldots, p-1$, define*

$$h_{bnpj}(y_1, y_2, \ldots, y_p)$$
$$= \sum_{t_1=1}^{n}\sum_{t_2=1}^{n}\cdots\sum_{t_p=1}^{n} E(I_{t_1,y_1,b}I_{t_2,y_2,b}\cdots I_{t_p,y_p,b})I(t_1\neq t_2)I(t_2\neq t_3)\cdots I(t_j\neq t_{j+1}),$$

and for $b = 1, 2$ and $p \geq 1$, define

$$h_{bnp0}(y_1, y_2, \ldots, y_p) = \sum_{t_1=1}^{n}\sum_{t_2=1}^{n}\cdots\sum_{t_p=1}^{n} E(I_{t_1,y_1,b}I_{t_2,y_2,b}\cdots I_{t_p,y_p,b}).$$

Then $H_{bpj}(y_1, y_2, \ldots, y_p) = \lim_{n\to\infty} h_{bnpj}(y_1, y_2, \ldots, y_p)$ is well-defined for $b = 1, 2$, $p \geq 1$ and $j = 0, 1, \ldots, p-1$, and as a consequence,

$$\mu_p(y_1, y_2, \ldots, y_p) = H_{1p0}(y_1, \ldots, y_p) = \lim_{n\to\infty} h_{1np0}(y_1, y_2, \ldots, y_p)$$

and

$$\nu_p(y_1, y_2, \ldots, y_p) = H_{2p0}(y_1, \ldots, y_p) = \lim_{n\to\infty} h_{2np0}(y_1, y_2, \ldots, y_p)$$

are well-defined for $p \geq 1$. Also, for $b = 1, 2$, $p \geq 2$, $j = 0, 1, \ldots, p-2$ and $0 \leq y_1 \leq y_2 \leq \cdots \leq y_p$,

$$H_{bpj}(y_1, y_2, \ldots, y_p) = H_{b,p-1,j}(y_1, \ldots, y_{j+1}, y_{j+3}, \ldots, y_p) + H_{b,p,j+1}(y_1, y_2, \ldots, y_p). \quad (29)$$

Proof of Lemma 14: For $b = 1, 2$, $p \geq 3$ and $j = 1, 2, \ldots, p-2$, we can write

$$h_{bnpj}(y_1, y_2, \ldots, y_p)$$
$$= \sum_{t_1=1}^{n}\sum_{t_2=1}^{n}\cdots\sum_{t_p=1}^{n} E(I_{t_1,y_1,b}I_{t_2,y_2,b}\cdots I_{t_p,y_p,b})I(t_1 \neq t_2)\cdots I(t_j \neq t_{j+1})$$

$$= \sum_{t_1=1}^{n}\sum_{t_2=1}^{n}\cdots\sum_{t_p=1}^{n} E(I_{t_1,y_1,b}I_{t_2,y_2,b}\cdots I_{t_p,y_p,b})I(t_1 \neq t_2)\cdots I(t_j \neq t_{j+1})(I(t_{j+1} = t_{j+2})+I(t_{j+1} \neq t_{j+2}))$$
$$= \sum_{t_1=1}^{n}\sum_{t_2=1}^{n}\cdots\sum_{t_{p-1}=1}^{n} E(I_{t_1,y_1,b}\cdots I_{t_j,y_j,b}I_{t_{j+1},y_{j+1},b}I_{t_{j+2},y_{j+3},b}\cdots I_{t_{p-1},y_p,b})I(t_1 \neq t_2)\cdots I(t_j \neq t_{j+1})$$
$$+ \sum_{t_1=1}^{n}\sum_{t_2=1}^{n}\cdots\sum_{t_p=1}^{n} E(I_{t_1,y_1,b}I_{t_2,y_2,b}\cdots I_{t_p,y_p,b})I(t_1 \neq t_2)\cdots I(t_j \neq t_{j+1})I(t_{j+1} \neq t_{j+2})$$

$$= h_{b,n,p-1,j}(y_1, y_2, \ldots, y_{j+1}, y_{j+3}, \ldots, y_p) + h_{b,n,p,j+1}(y_1, y_2, \cdots, y_p), \tag{30}$$

where the third equality follows from the fact that $I_{t_{j+1},y_{j+1},b}I_{t_{j+2},y_{j+2},b} = I_{t_{j+1},y_{j+1},b}I_{t_{j+1},y_{j+2},b} = I_{t_{j+1},\min(y_{j+1},y_{j+2}),b} = I_{t_{j+1},y_{j+1},b}$ when $t_{j+1} = t_{j+2}$ and from relabeling the summation indices. This shows the result of Equation (30) for $b = 1, 2$, $p \geq 3$ and $j = 1, 2, \ldots, p-2$. Similar to the argument in the proof of Lemma 7, for $b = 1, 2$, $p \geq 2$ and $j = 0$, we also have $h_{bnp0}(y_1, \ldots, y_p) = h_{b,n,p-1,0}(y_1, y_3 \ldots, y_p) + h_{bnp1}(y_1, \ldots, y_p)$. Therefore, we conclude that the result of Equation (30) holds for $b = 1, 2$, $p \geq 2$, and $j = 0, \ldots, p-2$. It follows from Lemma 13 that for $b = 1, 2$,

$$H_{b,p,p-1}(y_1, y_2, \ldots, y_p) = \lim_{n\to\infty} h_{b,n,p,p-1}(y_1, y_2, \ldots, y_p)$$

exists for $p \geq 1$. To show that $H_{bpj}(y_1, y_2, \ldots, y_p) = \lim_{n\to\infty} h_{bnpj}(y_1, y_2, \ldots, y_p)$ exists for $b = 1, 2$, $p \geq 2$ and $j = 0, \ldots, p-2$, note that by setting $j = p-2$ in Equation (30), we now have for $b = 1, 2$ and $p \geq 2$

$$h_{b,n,p,p-2}(y_1, y_2, \ldots, y_p) = h_{b,n,p-1,p-2}(y_1, y_2, \ldots, y_{p-1}) + h_{b,n,p,p-1}(y_1, y_2, \ldots, y_p).$$

By taking limits, it now follows that $H_{b,p,p-2}(y_1, y_2, \ldots, y_{p-1})$ exists because

$$H_{b,p,p-2}(y_1, y_2, \ldots, y_p) = H_{b,p-1,p-2}(y_1, y_2, \ldots, y_{p-1}) + H_{b,p,p-1}(y_1, y_2, \ldots, y_p).$$

Repeating this argument for $j = p - 3$, then $p - 4$ etc. until $j = 0$ then shows that $\lim_{n\to\infty} h_{bnpj}(y_1, y_2, \ldots, y_p) = H_{bpj}(y_1, \ldots, y_p)$ exists for $b = 1, 2$, $p \geq 2$ and $j = 0, \ldots, p - 2$. Therefore, the existence of $H_{bpj}(y_1, \ldots, y_p)$ is now shown for $b = 1, 2$, $p \geq 1$, and $j = 0, \ldots, p - 1$. The result of Equation (29) now follows by taking the limit as $n \to \infty$ in Equation (30). □

Lemma 15. *Assume Assumption 1 holds. Then for the* $\mu_p(y_1, y_2, \ldots, y_p)$ *sequence defined in Lemma 14, we have for* $p \geq 2$ *and* $y_1 \leq y_2 \leq \cdots \leq y_p$, $\mu_p(y_1, y_2, \ldots, y_p) \leq (2p)^p \max(1, y_p^p)$. *For the* $\nu_p(y_1, y_2, \ldots, y_p)$ *sequence defined in Lemma 14, we have for* $p \geq 2$ *and* $y_1 \leq y_2 \leq \cdots \leq y_p$, $\nu_p(y_1, y_2, \ldots, y_p) \leq 2^{2p} p^p \max(1, y_p^p)$.

Proof of Lemma 15: Note that $H_{bpj}(y_1, y_2, \ldots, y_p)$ is increasing in each argument y_j for $j = 1, 2, \ldots, p$. Therefore since $y_1 \leq y_2 \leq \ldots \leq y_p$

$$H_{1pj}(y_1, y_2, \ldots, y_p) \leq H_{1pj}(y_p, \ldots, y_p) \leq (2p)^p \max(1, y_p^p),$$

where the last inequality follows from Lemma 10 and setting $y = y_p$. The argument for $\nu_p(y_1, y_2, \ldots, y_p)$ is analogous. □

Lemma 16. *Assume Assumption 1 holds. Define*

$$X_{n1} = (\sum_{t=1}^{n} I_{t,y_1,1}, \sum_{t=1}^{n} I_{t,y_2,1}, \ldots, \sum_{t=1}^{n} I_{t,y_m,1})'$$

and

$$X_{n2} = (\sum_{t=1}^{n} I_{t,y_1,2}, \sum_{t=1}^{n} I_{t,y_2,2}, \ldots, \sum_{t=1}^{n} I_{t,y_m,2})'.$$

Then for all $p > 1$, $\mu_{\lambda p} = \lim_{n\to\infty} E(\lambda' X_{n1})^p$ *and* $\nu_{\lambda p} = \lim_{n\to\infty} E(\lambda' X_{n2})^p$ *are well-defined.*

Proof of Lemma 16: For the first case, by using the definition of $\mu_p(y_1, y_2, \ldots, y_p)$ in Lemma 14, we write that

$$\lim_{n\to\infty} E(\lambda' X_{n1})^p$$

$$= \sum_{j_1=1}^{m}\sum_{j_2=1}^{m}\cdots\sum_{j_p=1}^{m} \lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_p} \lim_{n\to\infty}\sum_{t_1=1}^{n}\sum_{t_2=1}^{n}\cdots\sum_{t_p=1}^{n} E(I_{t_1,y_{j_1},1}I_{t_2,y_{j_2},1}\cdots I_{t_p,y_{j_p},1})$$

$$= \sum_{j_1=1}^{m}\sum_{j_2=1}^{m}\cdots\sum_{j_p=1}^{m} \lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_p}\mu_p(y_{j_1}, y_{j_2}, \ldots, y_{j_p}). \tag{31}$$

Since $\mu_p(y_{j_1}, y_{j_2}, \ldots, y_{j_p})$ is well-defined by Lemma 14, $\lim_{n\to\infty} E(\lambda' X_{n1})^p$ is also well-defined. The second case is analogous, but uses the definition of $\nu_p(y_1, y_2, \ldots, y_p)$ instead of $\mu_p(y_1, y_2, \ldots, y_p)$ from Lemma 14. □

Lemma 17. *Assume Assumption 1 holds, and let $\mu_{\lambda p}$ and $\nu_{\lambda p}$ be as defined in Lemma 16. Then for $p \geq 2$,*

$$|\mu_{\lambda p}| \leq (2mp)^p \max_{1\leq i\leq m} |\lambda_i|^p \max(1, y_p^p)$$

and

$$|\nu_{\lambda p}| \leq 2^{2p}(mp)^p \max_{1\leq i\leq m} |\lambda_i|^p \max(1, y_p^p).$$

Proof of Lemma 17: Since $\mu_p(y_1, \ldots, y_p) \leq (2p)^p \max(1, y_p^p)$ by Lemma 15, it follows from Equation (31) that

$$|\mu_{\lambda p}| \leq \sum_{j_1=1}^{m}\sum_{j_2=1}^{m}\cdots\sum_{j_p=1}^{m} |\lambda_{j_1}||\lambda_{j_2}|\ldots|\lambda_{j_p}|\mu_p(y_{j_1}, y_{j_2}, \ldots, y_{j_p})$$

$$\le \sum_{j_1=1}^{m}\sum_{j_2=1}^{m}\cdots\sum_{j_p=1}^{m} \max_{1\le i\le m} |\lambda_i|^p (2p)^p \max(1, y_p^p)$$
$$= (2mp)^p \max_{1\le i\le m} |\lambda_i|^p \max(1, y_p^p).$$

The second part is shown analogously, except that the upper bound for $\nu_p(y_1, \ldots, y_p)$ from Lemma 15 is now used. □

We now provide the proof of Lemma 2.

Proof of Lemma 2: We first apply Lemma 11 to

$$X_{n1} = (\sum_{t=1}^{n} I_{t,y_1,1}, \sum_{t=1}^{n} I_{t,y_2,1}, \ldots, \sum_{t=1}^{n} I_{t,y_m,1})'$$

for part 1 of Lemma 2, and then to

$$X_{n2} = (\sum_{t=1}^{n} I_{t,y_1,2}, \sum_{t=1}^{n} I_{t,y_2,2}, \ldots, \sum_{t=1}^{n} I_{t,y_m,2})'$$

for part 2 of Lemma 2. Condition (1) of Lemma 11 holds because $E(\lambda' X_{n1})^p$ converges to $\mu_{\lambda p}$ by Lemma 16. Condition (2) of Lemma 11 follows because $\sum_{p=1}^{\infty} \mu_{\lambda,2p}^{-1/(2p)} = \infty$ because for $p \ge 2$, $\mu_{\lambda,2p} \le (4mp)^{2p} \max_{1\le i\le m}(1, \lambda_i^{2p}) \max(1, y_p^{2p})$ by Lemma 17. Therefore, the proof of the result for part 1 is now complete. The proof of part 2 of Lemma 2 is analogous. □

Proof of Theorem 2

Theorem 2 relies on Lemma 18.

Lemma 18. *For all* $(y_1, \ldots, y_m)' \in \mathbb{R}^m$ *and* $(y_1', \ldots, y_m')' \in \mathbb{R}^m$,

$$|P(Y_{ni} \le y_i \text{ for } i = 1, \ldots, m) - P(Y_{ni} \le y_i' \text{ for } i = 1, \ldots, m)|$$
$$\le \sum_{i=1}^{m} |P(Y_{ni} \le y_i) - P(Y_{ni} \le y_i')|.$$

Proof of Lemma 18: The result follows because

$$
\begin{aligned}
&|P(Y_{ni} \le y_i \text{ for } i = 1, \ldots, m) - P(Y_{ni} \le y_i' \text{ for } i = 1, \ldots, m)| \\
&= |E \prod_{i=1}^{m} (I(Y_{ni} \le y_i) - I(Y_{ni} \le y_i'))| \\
&\le E \prod_{i=1}^{m} |I(Y_{ni} \le y_i) - I(Y_{ni} \le y_i')| \\
&\le E \sum_{i=1}^{m} |I(Y_{ni} \le y_i) - I(Y_{ni} \le y_i')| \\
&= \sum_{i=1}^{m} EI(\min(y_i, y_i') \le Y_{ni} \le \max(y_i, y_i')) \\
&= \sum_{i=1}^{m} (P(Y_{ni} \le \max(y_i, y_i')) - P(Y_{ni} \le \min(y_i, y_i'))) \\
&= \sum_{i=1}^{m} |P(Y_{ni} \le y_i) - P(Y_{ni} \le y_i')|.
\end{aligned}
$$

□

We can now complete the proof of Theorem 2.

Proof of Theorem 2: We first show the first part of Theorem 2. Under Assumption 1, $(\sum_{t=1}^{n} I(0 \le x_t \le y_1 n^{-1/2}), \sum_{t=1}^{n} I(0 \le x_t \le y_2 n^{-1/2}), \ldots, \sum_{t=1}^{n} I(0 \le x_t \le y_m n^{-1/2}))'$ converges in distribution to $(R(y_1), R(y_2), \ldots, R(y_m))'$ by Lemma 2. We have

$$
P(Y_{ni} \le y_i \text{ for } i = 1, \ldots, m) = P(\sum_{t=1}^{n} I(0 < x_t \le y_i n^{-1/2}) \ge i - 1/2 \text{ for } i = 1, \ldots, m)
$$

and the last expression converges to some function $L(y_1, \ldots, y_m)$ since $(1/2, 3/2, \ldots, m-1/2)$ is a continuity point of the distribution of $(R(y_1), \ldots, R(y_m))'$. By Lemma 18, we have

$$
|L(y_1, \ldots, y_m) - L(y_1', \ldots, y_m')|
$$

$$= \lim_{n\to\infty} |P(Y_{ni} \le y_i \text{ for } i = 1, \ldots, m) - P(Y_{ni} \le y_i' \text{ for } i = 1, \ldots, m)|$$

$$\le \sum_{i=1}^{m} \lim_{n\to\infty} |P(Y_{ni} \le y_i) - P(Y_{ni} \le y_i')|$$

$$\le \sum_{i=1}^{m} \lim_{n\to\infty} P(\exists t \in \{1, \ldots, n\} : \min(y_i, y_i') < n^{1/2} x_t \le \max(y_i, y_i'))$$

$$\le C \sum_{i=1}^{m} |y_i - y_i'|, \tag{32}$$

where the last inequality follows from the reasoning of Equation (27). To show that $(Y_{n1}^{-1}, \ldots, Y_{nm}^{-1})'$ converges in distribution, note that if $z_i > 0$ for all i,

$$P(Y_{ni}^{-1} \le z_i \text{ for } i = 1, \ldots, m) = P(Y_{ni} \ge z_i^{-1} \text{ for } i = 1, \ldots, m)$$

converges as $n \to \infty$ and is continuous at $(z_1, \ldots, z_L)'$. Also, using the reasoning of Equation (32),

$$\limsup_{n\to\infty} |P(Y_{ni}^{-1} \le z_i \text{ for } i = 1, \ldots, m) - 1|$$

$$= \limsup_{n\to\infty} |P(Y_{ni} \ge z_i^{-1} \text{ for } i = 1, \ldots, m) - P(Y_{ni} \ge 0 \text{ for } i = 1, \ldots, m)|$$

$$\le \limsup_{n\to\infty} \sum_{i=1}^{m} P(0 \le Y_{ni} \le z_i^{-1}) \le C \sum_{i=1}^{m} z_i^{-1},$$

which converges to 0 if $z_i \to \infty$ for all i. Therefore, $\lim_{n\to\infty} P(Y_{ni}^{-1} \le z_i \text{ for } i = 1, \ldots, m)$ converges to 1 if $z_i \to \infty$ for all i. Together with the observation that $P(Y_{ni}^{-1} \le z_i \text{ for } i = 1, \ldots, m) = 0$ if $z_i < 0$, this implies that $P(Y_{n1}^{-1} \le z_1, \ldots, Y_{nm}^{-1} \le z_m)$ converges to a well-defined limit distribution. Therefore, $(Y_{n1}^{-1}, \ldots, Y_{nm}^{-1})'$ converges in distribution. The second part of the theorem is proven analogously.

□

Proof of Theorem 3

Lemma 19 is key to the proof of Theorem 3. The proof of Lemma 19 is taken from Pötscher (2013, Theorem 1) and is stated here for completeness.

Lemma 19. *Assume Assumption 1 holds. Then for $q > 1$,*

$$n^{-q/2}\sum_{t=1}^{n} x_t^{-q} I(x_t > 0) = O_p(1)$$

and

$$n^{-q/2}\sum_{t=1}^{n} |x_t|^{-q} = O_p(1).$$

Proof of Lemma 19: First note that, for all $\delta > 0$,

$$n^{-q/2}\sum_{t=1}^{n} x_t^{-q} I(x_t > 0) = n^{-q/2}\sum_{t=1}^{n} x_t^{-q} I(0 < x_t \leq \delta n^{-1/2}) + n^{-q/2}\sum_{t=1}^{n} x_t^{-q} I(x_t > \delta n^{-1/2}).$$

Now

$$En^{-q/2}\sum_{t=1}^{n} x_t^{-q} I(x_t > \delta n^{-1/2}) = n^{-q/2}\sum_{t=1}^{n} t^{-q/2} E(t^{-1/2}x_t)^{-q} I(t^{-1/2}x_t > \delta n^{-1/2}t^{-1/2})$$

$$= n^{-q/2}\sum_{t=1}^{n} t^{-q/2}\int_{t^{-1/2}n^{-1/2}\delta}^{\infty} x^{-q} f_t(x)dx$$

$$\leq \sup_{t\geq 1, x\in\mathbb{R}} f_t(x) n^{-q/2}\sum_{t=1}^{n} t^{-q/2}\int_{t^{-1/2}n^{-1/2}\delta}^{\infty} x^{-q}dx$$

$$= \sup_{t\geq 1, x\in\mathbb{R}} f_t(x) n^{-q/2}\sum_{t=1}^{n} t^{-q/2}(1/(1-q))[x^{-q+1}]_{t^{-1/2}n^{-1/2}\delta}^{\infty}$$

$$= \sup_{t\geq 1, x\in\mathbb{R}} f_t(x) n^{-q/2}\sum_{t=1}^{n} t^{-q/2}(1/(q-1))(t^{-1/2}n^{-1/2}\delta)^{-q+1}$$

$$= \sup_{t\geq 1, x\in\mathbb{R}} f_t(x) \sum_{t=1}^{n} (1/(q-1)) t^{-1/2} n^{-1/2} \delta^{-q+1}.$$

In addition, for all $\delta > 0$,

$$P(n^{-q/2} \sum_{t=1}^{n} x_t^{-q} I(0 < x_t < \delta n^{-1/2}) > 0)$$

$$= P(\exists t \in \{1, \ldots, n\} : 0 < x_t < \delta n^{-1/2}) \leq \sum_{t=1}^{n} P(0 < x_t < \delta n^{-1/2})$$

$$= \sum_{t=1}^{n} \int_0^{\delta n^{-1/2} t^{-1/2}} f_t(x) dx$$

$$\leq \sup_{t\geq 1, x\in\mathbb{R}} f_t(x) \sum_{t=1}^{n} \delta n^{-1/2} t^{-1/2}.$$

Therefore,

$$\limsup_{n\to\infty} P(n^{-q/2} \sum_{t=1}^{n} x_t^{-q} I(x_t > 0) > K)$$

$$\leq \limsup_{n\to\infty} P(n^{-q/2} \sum_{t=1}^{n} x_t^{-q} I(0 < x_t < \delta n^{-1/2}) > 0)$$

$$+ \limsup_{n\to\infty} P(n^{-q/2} \sum_{t=1}^{n} x_t^{-q} I(x_t > \delta n^{-1/2}) > K)$$

$$\leq \sup_{t\geq 1, x\in\mathbb{R}} f_t(x) \limsup_{n\to\infty} \sum_{t=1}^{n} \delta n^{-1/2} t^{-1/2} + K^{-1} \sup_{t\geq 1, x\in\mathbb{R}} f_t(x) \limsup_{n\to\infty} \sum_{t=1}^{n} (1/(q-1)) t^{-1/2} n^{-1/2} \delta^{-q+1}$$

$$\leq C_1 \delta + C_2 K^{-1} \delta^{-q+1}$$

for constants C_1 and C_2 independent of n, and the second inequality follows from the Markov inequality. Therefore,

$$\limsup_{K\to\infty} \limsup_{n\to\infty} P(n^{-q/2} \sum_{t=1}^{n} x_t^{-q} I(x_t > 0) > K) \leq C_1 \delta,$$

and since δ was arbitrary, the result now follows. For $n^{-q/2} \sum_{t=1}^{n} |x_t|^{-q}$, the same reasoning can be followed. □

We are now ready to prove the main result of the paper.

Proof of Theorem 3: Define

$$X_n = n^{-q/2}\sum_{t=1}^{n} x_t^{-q} I(x_t \geq 0);$$

$$M_n(K) = \sum_{t=1}^{n} I(0 \leq x_t \leq Kn^{-1/2});$$

$$X_{nK}^{(1)} = n^{-q/2}\sum_{t=1}^{n} x_t^{-q} I(0 \leq x_t \leq Kn^{-1/2}) = \sum_{t=1}^{M_n(K)} Y_{nt}^{-q} I(0 \leq Y_{nt} \leq K);$$

$$X_{nKm}^{(2)} = \sum_{t=1}^{\min(m,M_n(K))} Y_{nt}^{-q} I(0 \leq Y_{nt} \leq K);$$

$$X_{nm}^{(3)} = \sum_{t=1}^{\min(m,M_n(\infty))} Y_{nt}^{-q};$$

$$X_m^{(4)} = \sum_{t=1}^{m} Y_t^{-q};$$

and consider the Laplace transform $E\exp(-rX_n)$ for $r > 0$. We will first show that this Laplace transform converges by considering $E\exp(-rX_n) - E\exp(-rX_{nK}^{(1)})$, $E\exp(-rX_{nK}^{(1)}) - E\exp(-rX_{nKm}^{(2)})$, $E\exp(-rX_{nKm}^{(2)}) - E\exp(-rX_{nm}^{(3)})$, and $E\exp(-rX_{nm}^{(3)}) - E\exp(-rX_m^{(4)})$. For the first term, note that for all $r > 0$, because $|\exp(-x) - \exp(-y)| \leq |x - y|$ for $x, y \geq 0$,

$$\limsup_{n\to\infty} |E\exp(-rX_n) - E\exp(-rX_{nK}^{(1)})| \leq r \limsup_{n\to\infty} E|X_n - X_{nK}^{(1)}|$$

$$\leq r \limsup_{n\to\infty} n^{-q/2}\sum_{t=1}^{n} t^{-q/2} E(t^{-1/2}x_t)^{-q} I(t^{-1/2}x_t > t^{-1/2}n^{-1/2}K)$$

$$= r \limsup_{n\to\infty} n^{-q/2}\sum_{t=1}^{n} t^{-q/2}\int_{-\infty}^{\infty} f_t(x) x^{-q} I(x > t^{-1/2}n^{-1/2}K)dx$$

$$\leq r \sup_{t\geq 1, x\in\mathbb{R}} f_t(x) \limsup_{n\to\infty} n^{-q/2} \sum_{t=1}^{n} t^{-q/2}(q-1)^{-1}(t^{-1/2}n^{-1/2}K)^{1-q}$$

$$\leq CK^{1-q} \limsup_{n\to\infty} n^{-1/2} \sum_{t=1}^{n} t^{-1/2} = 2CK^{1-q}.$$

To deal with the second term, note that $X_{nK}^{(1)} \neq X_{nKm}^{(2)}$ if $m < M_n(K)$, and therefore

$$\limsup_{n\to\infty} |E\exp(-rX_{nK}^{(1)}) - E\exp(-rX_{nKm}^{(2)})|$$

$$= \limsup_{n\to\infty} |E(\exp(-rX_{nK}^{(1)}) - \exp(-rX_{nKm}^{(2)}))I(X_{nK}^{(1)} \neq X_{nKm}^{(2)})|$$

$$\leq 2 \limsup_{n\to\infty} P(X_{nK}^{(1)} \neq X_{nKm}^{(2)})$$

$$\leq 2 \limsup_{n\to\infty} P(M_n(K) > m)$$

$$\leq 2m^{-1} \limsup_{n\to\infty} E \sum_{t=1}^{n} I(0 \leq x_t \leq Kn^{-1/2})$$

$$\leq 2m^{-1} \limsup_{n\to\infty} \sum_{t=1}^{n} (F_t(t^{-1/2}Kn^{-1/2}) - F_t(0)) \leq Cm^{-1}K, \tag{33}$$

where the third inequality is the Markov inequality and the last inequality follows from the mean value theorem and $\sup_{t\geq 1, x\in\mathbb{R}} f_t(x) < \infty$. Also, defining a summation over an empty index set as 0, we have

$$|X_{nKm}^{(2)} - X_{nm}^{(3)}|$$

$$= |\sum_{t=1}^{\min(m,M_n(K))} Y_{nt}^{-q} I(0 \leq Y_{nt} \leq K) - \sum_{t=1}^{\min(m,M_n(\infty))} Y_{nt}^{-q}|$$

$$= |\sum_{t=1}^{\min(m,M_n(K))} Y_{nt}^{-q} - \sum_{t=1}^{\min(m,M_n(\infty))} Y_{nt}^{-q}| = \sum_{t=\min(m,M_n(K))+1}^{\min(m,M_n(\infty))} Y_{nt}^{-q}$$

$$\leq \sum_{t=M_n(K)+1}^{M_n(\infty)} Y_{nt}^{-q} = \sum_{t=M_n(K)+1}^{M_n(\infty)} Y_{nt}^{-q} I(Y_{nt} > K)$$

$$\leq n^{-q/2} \sum_{t=1}^{n} x_t^{-q} I(x_t > n^{-1/2}K) = |X_n - X_{nK}^{(1)}|,$$

where the first inequality holds because if $m \leq M_n(K)$, then $m \leq M_n(K) \leq M_n(\infty)$ and the summation equals 0. Therefore we can follow the earlier argument for the first term to find

$$\limsup_{n\to\infty} |E\exp(-rX_{nKm}^{(2)}) - E\exp(-rX_{nm}^{(3)})| \leq 2CK^{1-q}.$$

Also, note that because the random walk is recurrent, $M_n(\infty)$ will exceed any m eventually, implying that eventually, $X_{nm}^{(3)} = \sum_{t=1}^{m} Y_{nt}^{-q}$. Therefore, because $(Y_{n1}^{-1}, \ldots, Y_{nm}^{-1})'$ converges in distribution to $(Y_1^{-1}, \ldots, Y_m^{-1})'$ by Theorem 2, it follows that $X_{nm}^{(3)}$ converges in distribution to $X_m^{(4)}$. Putting all these results together, we now find

$$\limsup_{n\to\infty} |E\exp(-rX_n) - E\exp(-rX_m^{(4)})| \leq 2CK^{1-q} + Cm^{-1}K + 2CK^{1-q} + 0.$$

Next, note that $\psi(r) = \lim_{m\to\infty} E\exp(-rX_m^{(4)})$ is well-defined because $X_m^{(4)}$ is increasing in m. Therefore for all $K > 0$ and $m \geq 1$

$$\limsup_{n\to\infty} |E\exp(-rX_n) - \psi(r)|$$

$$\leq \limsup_{n\to\infty} |E\exp(-rX_n) - E\exp(-rX_m^{(4)})| + |E\exp(-rX_m^{(4)}) - \psi(r)|$$

$$\leq Cm^{-1}K + 4CK^{1-q} + |E\exp(-rX_m^{(4)}) - \psi(r)|.$$

Now taking the limit first as $m \to \infty$ and then $K \to \infty$, it follows that $\lim_{n\to\infty} E\exp(-rX_n) = \psi(r)$. By Theorem 2 of Feller (1971) on page 431, X_n now converges in distribution if $\lim_{r\downarrow 0} \psi(r) = 1$. Note that

$$\lim_{r\downarrow 0} \limsup_{n\to\infty} |E\exp(-rX_n) - 1|$$

$$\leq \limsup_{K\to\infty} \lim_{r\downarrow 0} \limsup_{n\to\infty} |E(\exp(-rX_n) - 1)(I(|X_n| \leq K) + I(|X_n| > K))|$$

$$\leq \limsup_{K\to\infty} \lim_{r\downarrow 0} \limsup_{n\to\infty} (|r|E|X_n|I(|X_n| \leq K) + P(|X_n| > K)),$$

and therefore it suffices to show

$$\limsup_{K\to\infty} \limsup_{n\to\infty} P(|X_n| > K) = 0,$$

that is, it suffices to show $X_n = O_p(1)$, which follows from Lemma 19.

To show that $n^{-q/2}\sum_{t=1}^{n}|x_t|^{-q} \xrightarrow{d} \sum_{t=1}^{\infty} Z_t^{-q}$, we can reason analogously by defining

$$X_n = n^{-q/2}\sum_{t=1}^{n}|x_t|^{-q};$$

$$M_n(K) = \sum_{t=1}^{n} I(|x_t| \leq Kn^{-1/2});$$

$$X_{nK}^{(1)} = n^{-q/2}\sum_{t=1}^{n}|x_t|^{-q}I(|x_t| \leq n^{-1/2}K) = \sum_{t=1}^{M_n(K)} Z_{nt}^{-q}I(Z_{nt} \leq K);$$

$$X_{nKm}^{(2)} = \sum_{t=1}^{\min(m, M_n(K))} Z_{nt}^{-q}I(Z_{nt} \leq K);$$

$$X_{nm}^{(3)} = \sum_{t=1}^{\min(m,n)} Z_{nt}^{-q};$$

$$X_m^{(4)} = \sum_{t=1}^{m} Z_t^{-q}.$$

□

References

Akonom, J. (1993): "Comportement asymptotique du temps d'occupation du processus des sommes partielles," *Annales de l'I.H.P. Probabilités et Statistiques*, 29, 57–81.

BERKES, I. AND L. HORVÁTH (2006): "Convergence of integral functionals of stochastic processes," *Econometric Theory*, 22, 304–322.

BORODIN, A. N. AND I. A. IBRAGIMOV (1995): *Limit theorems for functionals of random walks*, vol. 195, Proceedings of the Steklov Institute of Mathematics.

CHRISTOPEIT, N. (2009): "Weak convergence of nonlinear transformations of integrated processes: the multivariate case," *Econometric Theory*, 25, 1180–1207.

CHUNG, K. L. (2001): *A course in probability theory*, Academic Press, 3 ed.

DE JONG, R. M. (2004): "Addendum to "Asymptotics for nonlinear transformations of integrated time series"," *Econometric Theory*, 20, 627–635.

DE JONG, R. M. AND C. H. WANG (2005): "Further results on the asymptotics for non-linear transformations of integrated time series," *Econometric Theory*, 21, 413–430.

FELLER, W. (1971): *An introduction to probability theory and its applications, Volume II*, John Wiley & Sons, Inc. New York, N.Y., 2 ed.

FRÉCHET, M. AND J. SHOHAT (1931): "A proof of the generalized second-limit theorem in the theory of probability," *Transactions of the American Mathematical Society*, 33, 533–543.

JEGANATHAN, P. (2004): "Convergence of functionals of sums of r.v.s to local times of fractional stable motions," *The Annals of Probability*, 32, 1771–1795.

LIN, G. D. (2017): "Recent developments on the moment problem," *Journal of Statistical Distributions and Applications*, 4, 1–17.

Michel, J. and R. M. de Jong (2020): "The sum of the reciprocal of the random walk," *Econometric Theory*, 36, 170–183.

Miller, K. S. and B. Ross (1993): *An introduction to the fractional calculus and fractional differential equations*, Wiley.

Park, J. Y. and P. C. B. Phillips (1999): "Asymptotics for nonlinear transformations of integrated time series," *Econometric Theory*, 15, 269–298.

Pötscher, B. M. (2004): "Nonlinear functions and convergence to Brownian motion: beyond the continuous mapping theorem," *Econometric Theory*, 20, 1–22.

——— (2013): "On the order of magnitude of sums of negative powers of integrated processes," *Econometric Theory*, 29, 642–658.

Van der Vaart, A. W. (2000): *Asymptotic statistics*, Cambridge University Press.