Appendix to

"The spectral analysis of the Hodrick-Prescott filter"

by

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and

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A The properties of the weights of the HP filter

The HP filter can be defined as

$$\hat{\tau}_{Tt} = \sum_{s=1}^{T} w_{Tts} y_s$$

where w_{Tts} represents the weights of the HP filter at time point s with respect to the time point t and is derived by de Jong and Sakarya (2016). The formula of the weights w_{Tts} is

$$w_{Tts} = f_{T\lambda}(t-s) + f_{T\lambda}(t+s-1)I(t+s-1<T)$$

$$+f_{T\lambda}(T)I(t+s-1=T) + f_{T\lambda}(2T-t-s+1)I(t+s-1>T)$$

$$+\xi_{T\lambda}g_{T\lambda}(t)g_{T\lambda}(s) + \phi_{T\lambda}g_{T\lambda}(T-t+1)g_{T\lambda}(s)$$

$$+\phi_{T\lambda}g_{T\lambda}(t)g_{T\lambda}(T-s+1) + \xi_{T\lambda}g_{T\lambda}(T-t+1)g_{T\lambda}(T-s+1)$$

$$= \sum_{p=1}^{8} w_{Tts}^{p},$$

where for $m \in \mathbb{Z}$, $\lambda \in [0,\infty)$, and $T \geq 1$,

$$f_{T\lambda}(m) = \frac{1}{2T} + \frac{(-1)^m}{2T(1+16\lambda)} + T^{-1}\sum_{j=2}^{T} \frac{\cos(\pi(j-1)m/T)}{1+16\lambda\sin(\pi(j-1)/(2T))^4}.$$

The formulas for the function $g_{T\lambda}(.)$, and the constants $\xi_{T\lambda}$ and $\phi_{T\lambda}$ are provided in de Jong and Sakarya (2016) on page 312.

Properties of the weights:

1. $f_{T\lambda}(t-s) = f_{T\lambda}(s-t)$, and $\sum_{p=2}^{8} w_{Tts}^{p} = \sum_{p=2}^{8} w_{Tst}^{p}$ for all $s,t = 1,2,\ldots,T$ which implies that $w_{Tts} = w_{Tst}$ for all $s,t = 1,2,\ldots,T$.

2. $|f_{T\lambda}(0)| < 1$, $|f_{T\lambda}(m)| \leq Cm^{-3}$, and $|g_{T\lambda}(m)| \leq Cm^{-3}$ for $m = 1,2,\ldots,T$ for some constant C not depending on T.

3. $\sum_{p=2}^{8} |w_{Tts}^{p}| \leq C_1 t^{-3} + C_2(T-t+1)^{-3}(s^{-3} + (T-s+1)^{-3})$ for $s,t = 1,2,\ldots,T$ and for some constants C_1 and C_2 not depending on T.

4. $\sup_{1\leq s\leq T} \sum_{p=2}^{8} |w_{Tts}^{p}| \leq C_1 t^{-3} + C_2(T-t+1)^{-3}$ for $t = 1,2,\ldots,T$ and for some constants C_1 and C_2 not depending on T.

5. $\lim_{T\to\infty} f_{T\lambda}(m) = f_{\lambda}(m)$ pointwise in $m \in \mathbb{Z}$, and $f_{\lambda}(m) = f_{\lambda}(-m)$ for all $m \in \mathbb{Z}$.

6. $\sum_{m=-\infty}^{\infty} f_{\lambda}(m) = 1$.

Property 3 follows from Property 2. It is obvious that w^2_{Tts}, w^3_{Tts}, w^5_{Tts}, and w^7_{Tts} are bounded by Ct^{-3}. It should be noted that w^4_{Tts} is also bounded by Ct^{-3} because

$$f_{T\lambda}(2T-t-s+1) = f_{T\lambda}(t+s-1),$$

which is due to $\cos(\pi(j-1)(2T-t-s+1)/T) = \cos(\pi(j-1)(t+s-1)/T)$ for $j = 2, 3, \ldots T$. It is also clear that the weights w^6_{Tts} and w^8_{Tts} are bounded by $C(T-t+1)^{-3}(s^{-3} + (T-s+1)^{-3})$.

Also, note that Property 6 is verified by Lemma 6.

B Mathematical Proofs

Lemma 1. *Let* $\{y_t\}_{t=-\infty}^{\infty}$ *be a weakly stationary process with* $\gamma_y(s-l) = cov(y_s, y_l)$ *for* $s, l \in \mathbb{Z}$, *and* $\sum_{h=-\infty}^{\infty} |\gamma_y(h)| < \infty$. *Then,*

$$\lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} \sum_{s=1}^{T} \sum_{l=1}^{T} w_{Tts} w_{T,t-k,l} \gamma_y(s-l) = \lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} \sum_{s=1}^{T} \sum_{l=1}^{T} f_{T\lambda}(t-s) f_{T\lambda}(t-k-l) \gamma_y(s-l).$$

Proof of Lemma 1. To prove the result, we make use of an identity for the weights; $w_{Tts} = f_{T\lambda}(t-s) + \sum_{p=2}^{8} w^p_{Tts}$, and write that

$$\begin{aligned}
&\lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} \sum_{s=1}^{T} \sum_{l=1}^{T} w_{Tts} w_{T,t-k,l} \gamma_y(s-l) \\
&= \lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} \sum_{s=1}^{T} \sum_{l=1}^{T} \gamma_y(s-l) f_{T\lambda}(t-s) f_{T\lambda}(t-k-l) \\
&\quad + \lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} \sum_{s=1}^{T} \sum_{l=1}^{T} \gamma_y(s-l) f_{T\lambda}(t-s) (\sum_{q=2}^{8} w^q_{T,t-k,l}) && (1) \\
&\quad + \lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} \sum_{s=1}^{T} \sum_{l=1}^{T} \gamma_y(s-l) (\sum_{p=2}^{8} w^p_{Tts}) f_{T\lambda}(t-k-l) && (2) \\
&\quad + \lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} \sum_{s=1}^{T} \sum_{l=1}^{T} \gamma_y(s-l) (\sum_{p=2}^{8} w^p_{Tts}) (\sum_{q=2}^{8} w^q_{T,t-k,l}). && (3)
\end{aligned}$$

We will show that the expressions in Equations (1)-(3) vanish to prove the result.

First, consider the expression in Equation (1), and note that

$$\begin{aligned}
&\sum_{s=1}^{T} \sum_{l=1}^{T} \gamma_y(s-l) f_{T\lambda}(t-s) (\sum_{q=2}^{8} w^q_{T,t-k,l}) \\
&\leq \sum_{s=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} |\gamma_y(s-l)| |f_{T\lambda}(t-s)| (\sum_{q=2}^{8} |w^q_{T,t-k,l}|) I(1 \leq s \leq T) I(1 \leq l \leq T) \\
&\leq C_1 \sum_{s=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} |\gamma_y(s-l)| |t-s|^{-3} (C_2(t-k)^{-3} + C_3(T-t+k+1)^{-3}) \\
&\quad \times I(1 \leq s \leq T) I(1 \leq l \leq T) I(t \neq s)
\end{aligned}$$

$$+ C_4 \sum_{l=-\infty}^{\infty} |\gamma_y(s-l)|(C_2(t-k)^{-3} + C_3(T-t+k+1)^{-3})$$
$$\times I(1 \le s \le T) I(1 \le l \le T) I(t = s),$$

where $I(\cdot)$ is the indicator function that takes value of 0 or 1, the last inequality follows from Properties 2 and 4 in Appendix A. Equivalently, the above expression can be expressed as

$$C_1 \sum_{s=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} |\gamma_y(l)||s|^{-3}(C_2(t-k)^{-3} + C_3(T-t+k+1)^{-3})$$
$$\times I(t-T \le s \le t-1) I(s-T \le l \le s-1) I(s \ne 0)$$
$$+ C_4 \sum_{l=-\infty}^{\infty} |\gamma_y(l)|(C_2(t-k)^{-3} + C_3(T-t+k+1)^{-3})$$
$$\times I(t-T \le s \le t-1) I(s-T \le l \le s-1) I(s = 0)$$
$$\le C((t-k)^{-3} + (T-t+k+1)^{-3}),$$

since $\sum_{l=-\infty}^{\infty} |\gamma_y(l)| < \infty$, and $\sum_{s=-\infty}^{\infty} |s|^{-3} < \infty$.

Therefore, we can conclude that

$$\lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} \sum_{s=1}^{T} \sum_{l=1}^{T} \gamma_y(s-l) f_{T\lambda}(t-s)(\sum_{q=2}^{8} w^q_{T,t-k,l})$$
$$\le C \lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} ((t-k)^{-3} + (T-t+k+1)^{-3}) = 0,$$

due to the Cesáro mean.

The discussions for Equations (2) and (3) follow a similar reasoning to the one above; therefore we skip it. □

Lemma 2. *Let* $\{y_t\}_{t=-\infty}^{\infty}$ *be a weakly stationary process with* $\gamma_y(s-l) = cov(y_s, y_l)$ *for* $s, l \in \mathbb{Z}$, *and* $\sum_{h=-\infty}^{\infty} |\gamma_y(h)| < \infty$. *Then,*

$$\lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} \sum_{s=1}^{T} \sum_{l=1}^{T} f_{T\lambda}(t-s) f_{T\lambda}(t-k-l)\gamma_y(s-l) = \sum_{j=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} f_\lambda(k+j) f_\lambda(h+j)\gamma_y(h).$$

Proof of Lemma 2. To prove the result, we first show that $|T^{-1} \sum_{t=k+1}^{T} \sum_{s=1}^{T} \sum_{l=1}^{T} f_{T\lambda}(t-s) f_{T\lambda}(t-k-l)\gamma_y(s-l)| < C$ for some $C < \infty$ which does not depend on T, then we will apply the dominated convergence theorem to reach the result. Note that

$$T^{-1} \sum_{t=k+1}^{T} \sum_{s=1}^{T} \sum_{l=1}^{T} f_{T\lambda}(t-s) f_{T\lambda}(t-k-l)\gamma_y(s-l)$$
$$= T^{-1} \sum_{t=k+1}^{T} \sum_{s=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{T\lambda}(t-s) f_{T\lambda}(t-k-l)\gamma_y(s-l) I(1 \le s \le T) I(1 \le l \le T)$$

$$
\begin{aligned}
=&T^{-1}\sum_{t=k+1}^{T}\sum_{j=-\infty}^{\infty}\sum_{h=-\infty}^{\infty} f_{T\lambda}(k+j)f_{T\lambda}(h+j)\gamma_y(h)\\
&\times I(1\le t-k-j\le T)I(h+1\le t-k-j\le T+h)\\
\le&T^{-1}\sum_{t=k+1}^{T}\sum_{j=-\infty}^{\infty}\sum_{h=-\infty}^{\infty} |\gamma_y(h)||f_{T\lambda}(k+j)||f_{T\lambda}(h+j)|\\
&\times I(1\le t-k-j\le T)I(h+1\le t-k-j\le T+h)\\
<&\infty,
\end{aligned}
$$

where the second equality is obtained by setting $t-s=k+j$ and $s-l=h$ and the last inequality follows from Property 2 and $\sum_{h=-\infty}^{\infty}|\gamma_y(h)|<\infty$. Note that $|f_{T\lambda}(m)|\le Cm^{-3}$ for $m=1,2,\dots,T$ where C does not depend on T. Therefore, we can apply the dominated convergence theorem. Next, we write that

$$
\begin{aligned}
&\lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T}\sum_{j=-\infty}^{\infty}\sum_{h=-\infty}^{\infty} f_{T\lambda}(k+j)f_{T\lambda}(h+j)\gamma_y(h)\\
&\times I(1\le t-k-j\le T)I(h+1\le t-k-j\le T+h)\\
=&\sum_{j=-\infty}^{\infty}\sum_{h=-\infty}^{\infty}\gamma_y(h)\lim_{T\to\infty}T^{-1}\sum_{t=k+1}^{T} f_{T\lambda}(k+j)f_{T\lambda}(h+j)\\
&\times I(1\le t-k-j\le T)I(h+1\le t-k-j\le T+h)\\
=&\sum_{j=-\infty}^{\infty}\sum_{h=-\infty}^{\infty} f_{\lambda}(k+j)f_{\lambda}(h+j)\gamma_y(h)\\
&\times\lim_{T\to\infty}(T+\min\{0,k+j,k+j+h\}-(k+1)-\max\{0,j,j+h\})/T\\
=&\sum_{j=-\infty}^{\infty}\sum_{h=-\infty}^{\infty} f_{\lambda}(k+j)f_{\lambda}(h+j)\gamma_y(h),
\end{aligned}
$$

where the second equality follows from Property 5 in Appendix A, and the last equality follows from

$$
\lim_{T\to\infty}(T+\min\{0,k+j,k+j+h\}-(k+1)-\max\{0,j,j+h\})/T=1.
$$

□

Lemma 3. *Let* $\{y_t\}_{t=-\infty}^{\infty}$ *be a weakly stationary process with* $\gamma_y(s-l)=cov(y_s,y_l)$ *for* $s,l\in\mathbb{Z}$, *and* $\sum_{h=-\infty}^{\infty}|\gamma_y(h)|<\infty$. *Then,*

$$
\lim_{T\to\infty}T^{-1}\sum_{t=k+1}^{T}\sum_{s=1}^{T} w_{Tts}\gamma_y(s-t+k)=\lim_{T\to\infty}T^{-1}\sum_{t=k+1}^{T}\sum_{s=1}^{T} f_{T\lambda}(t-s)\gamma_y(s-t+k).
$$

Proof of Lemma 3. We use an identity for the weights; $w_{Tts} = f_{T\lambda}(t-s) + \sum_{p=2}^{8} w_{Tts}^p$; therefore, we can write that

$$\begin{aligned}
&\lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} \sum_{s=1}^{T} w_{Tts}\gamma_y(s-t+k) \\
=&\lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} \sum_{s=1}^{T} f_{T\lambda}(t-s)\gamma_y(s-t+k) + \lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} \sum_{s=1}^{T} (\sum_{p=2}^{8} w_{Tts}^p)\gamma_y(s-t+k).
\end{aligned}$$

We will show that the second term of the expression above is zero. Note that

$$\begin{aligned}
&\sum_{s=1}^{T} (\sum_{p=2}^{8} w_{Tts}^p)\gamma_y(s-t+k) \leq \sum_{s=1}^{T} (\sum_{p=2}^{8} |w_{Tts}^p|)|\gamma_y(s-t+k)| \\
\leq& \sum_{s=-\infty}^{\infty} (C_1 t^{-3} + C_2(T-t+1)^{-3})|\gamma_y(s-t+k)| I(1 \leq s \leq T),
\end{aligned}$$

by Property 4 in Appendix A. The above expression can be rewritten as

$$\begin{aligned}
&\sum_{s=-\infty}^{\infty} (C_1 t^{-3} + C_2(T-t+1)^{-3})|\gamma_y(s)| I(1-t+k \leq s \leq T-t+k) \\
\leq&(C_3 t^{-3} + C_4(T-t+1)^{-3}),
\end{aligned}$$

since $\sum_{s=-\infty}^{\infty} |\gamma_y(s)| < \infty$. Therefore, we can conclude that

$$\begin{aligned}
&\lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} \sum_{s=1}^{T} (\sum_{p=2}^{8} w_{Tts}^p)\gamma_y(s-t+k) \\
\leq& \lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} (C_3 t^{-3} + C_4(T-t+1)^{-3}) = 0,
\end{aligned}$$

by the Cesáro mean. □

Lemma 4. *Let $\{y_t\}_{t=-\infty}^{\infty}$ be a weakly stationary process with $\gamma_y(s-l) = cov(y_s, y_l)$ for $s, l \in \mathbb{Z}$, and $\sum_{h=-\infty}^{\infty} |\gamma_y(h)| < \infty$. Then,*

$$\lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} \sum_{s=1}^{T} f_{T\lambda}(t-s)\gamma_y(t-k-s) = \sum_{h=-\infty}^{\infty} f_\lambda(h)\gamma_y(h-k).$$

Proof of Lemma 4. To prove the result, we first show that $|T^{-1} \sum_{t=k+1}^{T} \sum_{s=1}^{T} f_{T\lambda}(t-s)\gamma_y(t-k-s)| < C$ for some $C < \infty$ which does not depend on T, then we will apply the dominated convergence theorem to reach the result. Note that

$$\begin{aligned}
&T^{-1} \sum_{t=k+1}^{T} \sum_{s=1}^{T} f_{T\lambda}(t-s)\gamma_y(t-k-s) \\
\leq&T^{-1} \sum_{t=k+1}^{T} \sum_{s=1}^{T} |f_{T\lambda}(t-s)||\gamma_y(t-k-s)|
\end{aligned}$$

$$\le C_1 T^{-1}\sum_{t=k+1}^{T}\sum_{s=1}^{T}|t-s|^{-3}|\gamma_y(t-k-s)|I(t\neq s)$$
$$+ C_2 T^{-1}\sum_{t=k+1}^{T}\sum_{s=1}^{T}|\gamma_y(t-k-s)|I(t=s)$$
$$= C_1 T^{-1}\sum_{t=k+1}^{T}\sum_{h=-\infty}^{\infty}|h|^{-3}|\gamma_y(h-k)|I(t\neq s)I(1\le s\le T)$$
$$+ C_2|\gamma_y(k)|((T-1-\max\{0,k\})/T)$$
$$<\infty,$$

where the second inequality follows from Property 2, and the last inequality holds because $\sum_{h=-\infty}^{\infty}|\gamma_y(h)| < \infty$ and $(T-k)/T < 2$ for $k = -T+1,\dots,0,\dots,T-1$. Therefore, we can apply the dominated convergence theorem, and write that

$$\lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T}\sum_{s=1}^{T} f_{T\lambda}(t-s)\gamma_y(t-k-s)$$
$$= \lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T}\sum_{h=-\infty}^{\infty} f_{T\lambda}(h)\gamma_y(h-k)I(h+1\le t\le T+h)$$
$$= \sum_{h=-\infty}^{\infty}\gamma_y(h-k)\lim_{T\to\infty} f_{T\lambda}(h)T^{-1}\sum_{t=k+1}^{T} I(h+1\le t\le T+h)$$
$$= \sum_{h=-\infty}^{\infty} f_\lambda(h)\gamma_y(h-k)\lim_{T\to\infty}(T+\min\{0,h\}-1-\max\{k,h\})/T$$
$$= \sum_{h=-\infty}^{\infty} f_\lambda(h)\gamma_y(h-k),$$

we obtained the first equality by setting $h = t-s$, and the third equality follows from Property 5 in Appendix A. □

Lemma 5. *Let*

$$\gamma_\tau(k) = \lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T} cov(\hat\tau_{Tt},\hat\tau_{T,t-k}), \tag{4}$$
$$\gamma_c(k) = \lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T} cov(\hat c_{Tt},\hat c_{T,t-k}), \tag{5}$$

for $k\in\mathbb{Z}$ and

$$\hat\tau_{Tt} = \sum_{s=1}^{T} w_{Tts}y_s,$$

where $\hat c_{Tt} = y_t - \hat\tau_{Tt}$, and $\{y_t\}_{t=-\infty}^{\infty}$ is a weakly stationary process with $\gamma_y(h) = cov(y_t, y_{t-h})$ for $h\in\mathbb{Z}$. Then

$$\gamma_\tau(k) = \sum_{h=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\gamma_y(h)f_\lambda(h+j)f_\lambda(k+j)$$

$$\gamma_c(k) = \gamma_y(k) - 2\sum_{j=-\infty}^{\infty}\gamma_y(j)f_\lambda(j+k) + \sum_{h=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\gamma_y(h)f_\lambda(h+j)f_\lambda(k+j).$$

Proof of Lemma 5. We write that

$$\lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T} cov(\hat{\tau}_{Tt},\hat{\tau}_{T,t-k}) = \lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T} cov(\sum_{s=1}^{T} w_{Tts}y_s, \sum_{l=1}^{T} w_{T,t-k,l}y_l) \tag{6}$$

$$= \lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T}\sum_{s=1}^{T}\sum_{l=1}^{T} w_{Tts}w_{T,t-k,l}cov(y_s,y_l)$$

$$= \lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T}\sum_{s=1}^{T}\sum_{l=1}^{T} f_{T\lambda}(t-s)f_{T\lambda}(t-k-l)\gamma_y(s-l)$$

$$= \sum_{j=-\infty}^{\infty}\sum_{h=-\infty}^{\infty} f_\lambda(k+j)f_\lambda(h+j)\gamma_y(h),$$

where the third and last equalities are due to Lemmas 1 and 2, respectively.

Since $\hat{c}_{Tt} = y_t - \hat{\tau}_{Tt}$,

$$\lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T} cov(\hat{c}_{Tt},\hat{c}_{T,t-k})$$

$$= \lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T} cov(y_t,y_{t-k}) - \lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T} cov(\hat{\tau}_{Tt},y_{t-k}) \tag{7}$$

$$- \lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T} cov(\hat{\tau}_{T,t-k},y_t) + \lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T} cov(\hat{\tau}_{Tt},\hat{\tau}_{T,t-k}). \tag{8}$$

The first term in Equation (7) is $\gamma_y(k)$ by definition. Additionally, we have already derived the formula for the last term in Equation (8); therefore, it only remains to derive the other two terms.

By using the weighted average formulation of the HP filter, we have

$$\lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T} cov(\hat{\tau}_{Tt},y_{t-k}) = \lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T}\sum_{s=1}^{T} w_{Tts}cov(y_s,y_{t-k})$$

$$= \lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T}\sum_{s=1}^{T} f_{T\lambda}(t-s)\gamma_y(t-k-s) = \sum_{h=-\infty}^{\infty} f_\lambda(h)\gamma_y(h-k)$$

$$= \sum_{h=-\infty}^{\infty} f_\lambda(h+k)\gamma_y(h),$$

the second equality holds due to Lemma 3, and the last equality holds due to Lemma 4. Furthermore, the above result also implies that

$$\lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T} cov(\hat{\tau}_{T,t-k},y_t) = \sum_{l=-\infty}^{\infty} f_\lambda(l-k)\gamma_y(l) = \sum_{l=-\infty}^{\infty} f_\lambda(l+k)\gamma_y(l),$$

where the last equality holds since $f_\lambda(l-k) = f_\lambda(-l+k)$ by Property 5 in Appendix A, and $\gamma_y(l) = \gamma_y(-l)$. □

Lemma 6. *Let* $f_\lambda : \mathbb{Z} \to \mathbb{R}$ *such that*

$$f_\lambda(k) = \int_0^1 h_\lambda(\pi r)\cos(\pi r k)dr, \tag{9}$$

then

$$\sum_{k=-n}^{n} f_\lambda(k)e^{-i\omega k} \to h_\lambda(\omega),$$

uniformly on $\omega \in [-\pi, \pi]$ *as* $n \to \infty$ *where* $h_\lambda(\omega) = (1 + 16\lambda \sin(\omega/2)^4)^{-1}$.

Proof of Lemma 6. To show that

$$\sum_{k=-n}^{n} f_\lambda(k)e^{-i\omega k} \to h_\lambda(\omega),$$

uniformly on $[-\pi, \pi]$ as $n \to \infty$, we will use a set of Fourier series results. First note that $\{1, e^{-i\omega}, e^{-2i\omega}, \cdots\}$ is a system of orthogonal functions and the following is the Fourier series generated by $h_\lambda(\omega)$

$$h_\lambda(\omega) \sim \sum_{k=-\infty}^{\infty} f_\lambda(k)e^{-i\omega k} = f_\lambda(0) + 2\sum_{k=1}^{\infty} f_\lambda(k)\cos(\omega k), \tag{10}$$

since $f_\lambda(k)$ is an even function (i.e. $f_\lambda(k) = f_\lambda(-k)$) and Euler's formula[1]. The Fourier coefficients are

$$f_\lambda(k) = (1/2\pi)\int_{-\pi}^{\pi} h_\lambda(\omega)\cos(\omega k)d\omega, \tag{11}$$

for $k \in \mathbb{Z}_+$. The representation in Equation (10) is so-called Fourier Cosine Series.

Note that Equation (9) is equal to Equation (11) by a simple change of variable argument. If we set $\pi r = \omega$, then

$$f_\lambda(k) = \int_0^1 h_\lambda(\pi r)\cos(\pi r k)dr = (1/\pi)\int_0^\pi h_\lambda(\omega)\cos(\omega k)d\omega$$
$$= (1/2\pi)\int_{-\pi}^{\pi} h_\lambda(\omega)\cos(\omega k)d\omega,$$

the last equality follows from both $h_\lambda(\cdot)$ and cosine being even functions.

Now, we need to show that the Fourier series in Equation (10) converges to $h_\lambda(\omega)$ uniformly on $[-\pi, \pi]$. The sufficient conditions for the uniformity of the convergence of Fourier series are given in Brown and Churchill (2001)[2]. The sufficient conditions are satisfied because $h_\lambda(\omega)$ is a continuous function on $[-\pi, \pi]$ with $h_\lambda(-\pi) = h_\lambda(\pi)$ since it is an even function and its first derivative is also continuous on $(-\pi, \pi)$. Therefore, we can conclude that the Fourier series in Equation (10) converges to $h_\lambda(\omega)$ uniformly on $[-\pi, \pi]$. □

Lemma 7. *Let $\{u_t\}_{t=-\infty}^{\infty}$ be a weakly stationary process with the spectrum*

$$I_u(\omega) = (2\pi)^{-1} \sum_{h=-\infty}^{\infty} \gamma_u(h) e^{-i\omega h},$$

for $\omega \in [-\pi, \pi]$, then the spectrum of $\hat{\tau}_{Tt}$ and $\hat{c}_{Tt}$, obtained from u_t are,

$$I_\tau(\omega) = h_\lambda(\omega)^2 I_u(\omega)$$
$$I_c(\omega) = (1 - h_\lambda(\omega))^2 I_u(\omega),$$

where $h_\lambda(\omega) = (1 + 16\lambda \sin(\omega/2)^4)^{-1}$.

Proof of Lemma 7. The definition of the spectrum and Lemma 5 imply that

$$\begin{aligned}(2\pi)^{-1} \sum_{k=-\infty}^{\infty} \gamma_\tau(k) e^{-i\omega k} &= (2\pi)^{-1} \sum_{k=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \gamma_u(h) f_\lambda(h+j) f_\lambda(k+j) e^{-i\omega k} \\ &= (2\pi)^{-1} \sum_{h=-\infty}^{\infty} \gamma_u(h) \sum_{j=-\infty}^{\infty} f_\lambda(h+j) \sum_{k=-\infty}^{\infty} f_\lambda(k+j) e^{-i\omega k},\end{aligned}$$

where the last equality follows from Fubini's Theorem since

$$\begin{aligned}&(2\pi)^{-1} \sum_{k=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |\gamma_u(h)||f_\lambda(h+j)||f_\lambda(k+j)||e^{-i\omega k}| \\ \leq & C_1 \sum_{k=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |\gamma_u(h)||h+j|^{-3}|k+j|^{-3} I(h \neq j) I(k \neq j) \\ &+ C_2 \sum_{k=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |\gamma_u(h)||k+j|^{-3} I(h = j) I(k \neq j) \\ &+ C_3 \sum_{k=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |\gamma_u(h)||h+j|^{-3} I(h \neq j) I(k = j) \\ &+ C_4 \sum_{k=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |\gamma_u(h)| I(h = j) I(k = j) < \infty,\end{aligned}$$

by $\sum_{h=-\infty}^{\infty} |\gamma_u(h)| < \infty$.

Note that

$$\begin{aligned}&(2\pi)^{-1} \sum_{h=-\infty}^{\infty} \gamma_u(h) \sum_{j=-\infty}^{\infty} f_\lambda(h+j) \sum_{k=-\infty}^{\infty} f_\lambda(k+j) e^{-i\omega k} \\ =&(2\pi)^{-1} \sum_{h=-\infty}^{\infty} \gamma_u(h) \sum_{j=-\infty}^{\infty} f_\lambda(h+j) \sum_{k=-\infty}^{\infty} f_\lambda(k) e^{-i\omega(k-j)} \\ =&(2\pi)^{-1} h_\lambda(\omega) \sum_{h=-\infty}^{\infty} \gamma_u(h) \sum_{j=-\infty}^{\infty} f_\lambda(h+j) e^{i\omega j} \\ =&(2\pi)^{-1} h_\lambda(\omega)^2 \sum_{h=-\infty}^{\infty} \gamma_u(h) e^{-i\omega h} = h_\lambda(\omega)^2 I_u(\omega),\end{aligned} \tag{12}$$

by Property 5 and Lemma 6.

Next, we use the result of Lemma 5 and Lemma 6 to write that

$$
\begin{aligned}
&(2\pi)^{-1}\sum_{k=-\infty}^{\infty}\gamma_c(k)e^{-i\omega k}\\
=&(2\pi)^{-1}\sum_{k=-\infty}^{\infty}(\gamma_u(k)-2\sum_{j=-\infty}^{\infty}\gamma_u(j)f_\lambda(j+k)+\sum_{h=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\gamma_u(h)f_\lambda(h+j)f_\lambda(k+j))e^{-i\omega k}\\
=&(2\pi)^{-1}\sum_{k=-\infty}^{\infty}\gamma_u(k)e^{-i\omega k}-2(2\pi)^{-1}\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\gamma_u(j)f_\lambda(j+k)e^{-i\omega k}\\
&+(2\pi)^{-1}\sum_{k=-\infty}^{\infty}\sum_{h=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\gamma_u(h)f_\lambda(h+j)f_\lambda(k+j)e^{-i\omega k} \qquad (13)\\
=&I_u(\omega)-2h_\lambda(\omega)(2\pi)^{-1}\sum_{j=-\infty}^{\infty}\gamma_u(j)e^{i\omega j}+I_u(\omega)h_\lambda(\omega)^2\\
=&I_u(\omega)-2h_\lambda(\omega)I_u(\omega)+I_u(\omega)h_\lambda(\omega)^2\\
=&(1-h_\lambda(\omega))^2I_u(\omega),
\end{aligned}
$$

by Lemma 6. Note that we have derived the expression in Equation (13) above. □

Lemma 8. *Let* $\{\varepsilon_t\}_{t=-\infty}^{\infty}$ *be a weakly stationary process with* $\gamma_\varepsilon(i-j)=cov(\varepsilon_i,\varepsilon_j)$ *for* $i,j\in\mathbb{Z}$, *and* $\sum_{h=-\infty}^{\infty}|\gamma_\varepsilon(h)|<\infty$. *Then,*

$$
\begin{aligned}
&\lim_{T\to\infty}T^{-1}\sum_{t=k+1}^{T}\sum_{j=1}^{T}\sum_{i=1}^{T}\gamma_\varepsilon(i-j)\sum_{s=1}^{j-1}\sum_{l=1}^{i-1}w_{Tts}w_{T,t-k,l}I(2\le j\le t)I(2\le i\le t-k)\\
=&\lim_{T\to\infty}T^{-1}\sum_{t=k+1}^{T}\sum_{j=1}^{T}\sum_{i=1}^{T}\gamma_\varepsilon(i-j)\sum_{s=1}^{j-1}\sum_{l=1}^{i-1}f_{T\lambda}(t-s)f_{T\lambda}(t-k-l)I(2\le j\le t)I(2\le i\le t-k).
\end{aligned}
$$

Proof of Lemma 8. In order to prove the result, we use an identity of the weights given in Appendix A; $w_{Tts}=f_{T\lambda}(t-s)+\sum_{p=2}^{8}w_{Tts}^{p}$. Thus, we can write that

$$
\begin{aligned}
&\lim_{T\to\infty}T^{-1}\sum_{t=k+1}^{T}\sum_{j=1}^{T}\sum_{i=1}^{T}\gamma_\varepsilon(i-j)\sum_{s=1}^{j-1}\sum_{l=1}^{i-1}w_{Tts}w_{T,t-k,l}I(2\le j\le t)I(2\le i\le t-k)\\
=&\lim_{T\to\infty}T^{-1}\sum_{t=k+1}^{T}\sum_{j=1}^{T}\sum_{i=1}^{T}\gamma_\varepsilon(i-j)\sum_{s=1}^{j-1}\sum_{l=1}^{i-1}f_{T\lambda}(t-s)f_{T\lambda}(t-k-l)\\
&\times I(2\le j\le t)I(2\le i\le t-k)\\
&+\lim_{T\to\infty}T^{-1}\sum_{t=k+1}^{T}\sum_{j=1}^{T}\sum_{i=1}^{T}\gamma_\varepsilon(i-j)\sum_{s=1}^{j-1}\sum_{l=1}^{i-1}f_{T\lambda}(t-s)(\sum_{q=2}^{8}w_{T,t-k,l}^{q}) \qquad (14)\\
&\times I(2\le j\le t)I(2\le i\le t-k)\\
&+\lim_{T\to\infty}T^{-1}\sum_{t=k+1}^{T}\sum_{j=1}^{T}\sum_{i=1}^{T}\gamma_\varepsilon(i-j)\sum_{s=1}^{j-1}\sum_{l=1}^{i-1}(\sum_{p=2}^{8}w_{Tts}^{p})f_{T\lambda}(t-k-l) \qquad (15)\\
&\times I(2\le j\le t)I(2\le i\le t-k)
\end{aligned}
$$

$$+\lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T}\sum_{j=1}^{T}\sum_{i=1}^{T}\gamma_\varepsilon(i-j)\sum_{s=1}^{j-1}\sum_{l=1}^{i-1}(\sum_{p=2}^{8} w_{Tts}^{p})(\sum_{q=2}^{8} w_{T,t-k,l}^{q}) \tag{16}$$
$$\times I(2\le j\le t)I(2\le i\le t-k).$$

We will show that the expressions in Equation (14)-(16) vanish to zero.

First, consider the expression in Equation (14) and note that

$$\begin{aligned}
&\sum_{j=1}^{T}\sum_{i=1}^{T}\gamma_\varepsilon(i-j)\sum_{s=1}^{j-1}\sum_{l=1}^{i-1} f_{T\lambda}(t-s)(\sum_{q=2}^{8} w_{T,t-k,l}^{q})I(2\le j\le t)I(2\le i\le t-k)\\
\le&\sum_{j=1}^{T}\sum_{i=1}^{T}|\gamma_\varepsilon(i-j)|\sum_{s=t-j+1}^{t-1}\sum_{l=1}^{i-1}|f_{T\lambda}(s)|(\sum_{q=2}^{8}|w_{T,t-k,l}^{q}|)I(2\le j\le t)I(2\le i\le t-k)\\
\le& C\sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty}|\gamma_\varepsilon(i-j)|\sum_{s=t-j+1}^{t-1} s^{-3}I(2\le j\le t)I(2\le i\le t-k)\\
&\times\sum_{l=1}^{i-1}(C_1(t-k)^{-3}+C_2(T-t+k+1)^{-3}(l^{-3}+(T-l+1)^{-3}))\\
=& C\sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty}|\gamma_\varepsilon(i-j)|\sum_{s=t-j+1}^{t-1} s^{-3}I(2\le j\le t)I(2\le i\le t-k)\\
&\times(C_1(i-1)(t-k)^{-3}+C_2(T-t+k+1)^{-3}(\sum_{l=1}^{i-1}l^{-3}+\sum_{l=T-i+2}^{T}l^{-3}))\\
\le& C\sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty}|\gamma_\varepsilon(i-j)|\sum_{s=t-j+1}^{t-1} s^{-3}I(2\le j\le t)I(2\le i\le t-k)\\
&\times(C_1(t-k)^{-2}+C_2(T-t+k+1)^{-3}(\sum_{l=1}^{\infty}l^{-3}+\sum_{l=1}^{\infty}l^{-3})),
\end{aligned}$$

where the second inequality follows from Property 2 and 3 in Appendix A. We can apply a change of variables argument by setting $h=i-j$, $m=t-j$ and $m-h=t-i$, and rewrite the above expression as

$$\begin{aligned}
&(C_3(t-k)^{-2}+C_4(T-t+k+1)^{-3})\sum_{m=-\infty}^{\infty}\sum_{h=-\infty}^{\infty}|\gamma_\varepsilon(h)|\sum_{s=m+1}^{t-1}s^{-3}\\
&\times I(0\le m\le t-2)I(k\le m-h\le t-2)\\
\le&(C_3(t-k)^{-2}+C_4(T-t+k+1)^{-3})\sum_{m=-\infty}^{\infty}\sum_{s=m+1}^{\infty}s^{-3}I(0\le m\le t-2)\\
\le& C_5((t-k)^{-2}+(T-t+k+1)^{-3}),
\end{aligned}$$

where the first inequality follows from $\sum_{h=-\infty}^{\infty}|\gamma_\varepsilon(h)|<\infty$. Therefore, we can write that

$$\lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T}\sum_{j=1}^{T}\sum_{i=1}^{T}\gamma_\varepsilon(i-j)\sum_{s=1}^{j-1}\sum_{l=1}^{i-1} f_{T\lambda}(t-s)(\sum_{q=2}^{8} w_{T,t-k,l}^{q})I(2\le j\le t)I(2\le i\le t-k)$$

$$\leq C_5 \lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} ((t-k)^{-2} + (T-t+k+1)^{-3}) = 0.$$

The argument for the expression in Equation (15) follows from a similar discussion as above, so we skip it.

Next, consider the expression in Equation (16), and note that

$$\begin{aligned}
&\sum_{j=1}^{T}\sum_{i=1}^{T} \gamma_\varepsilon(i-j) \sum_{s=1}^{j-1}\sum_{l=1}^{i-1} (\sum_{p=2}^{8} w_{Tts}^{p})(\sum_{q=2}^{8} w_{T,t-k,l}^{q}) I(2 \leq j \leq t) I(2 \leq i \leq t-k) \\
\leq &\sum_{j=1}^{T}\sum_{i=1}^{T} |\gamma_\varepsilon(i-j)| \sum_{s=1}^{j-1}\sum_{l=1}^{i-1} (\sum_{p=2}^{8} |w_{Tts}^{p}|)(\sum_{q=2}^{8} |w_{T,t-k,l}^{q}|) I(2 \leq j \leq t) I(2 \leq i \leq t-k) \\
\leq &\sum_{j=1}^{T}\sum_{i=1}^{T} |\gamma_\varepsilon(i-j)| I(2 \leq j \leq t) I(2 \leq i \leq t-k) \\
&\times \sum_{s=1}^{j-1} (C_1 t^{-3} + C_2 (T-t+1)^{-3} (s^{-3} + (T-s+1)^{-3})) \\
&\times \sum_{l=1}^{i-1} (C_3 (t-k)^{-3} + C_4 (T-t+k+1)^{-3} (l^{-3} + (T-l+1)^{-3})) \\
\leq &\sum_{j=1}^{T}\sum_{i=1}^{T} |\gamma_\varepsilon(i-j)| I(2 \leq j \leq t) I(2 \leq i \leq t-k) \\
&\times (C_1 t^{-2} + C_2 (T-t+1)^{-3} (\sum_{s=1}^{t-1} s^{-3} + \sum_{s=T-t+2}^{T} s^{-3})) \\
&\times (C_3 (t-k)^{-2} + C_4 (T-t+k+1)^{-3} (\sum_{l=1}^{t-k-1} l^{-3} + \sum_{l=T-t+k+2}^{T} l^{-3})) \\
< &\sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty} |\gamma_\varepsilon(i-j)| I(2 \leq j \leq t) I(2 \leq i \leq t-k) \\
&\times (C_1 t^{-2} + C_5 (T-t+1)^{-3} ((t-1)^{-2} + T^{-2})) \\
&\times (C_3 (t-k)^{-2} + C_6 (T-t+k+1)^{-3}),
\end{aligned}$$

where the second inequality follows from Property 3 in Appendix A. We obtained the last inequality because $\sum_{s=1}^{t-1} s^{-3} < (t-1)^{-2}$, $\sum_{s=T-t+2}^{T} s^{-3} < \sum_{s=2}^{T} s^{-3} < T^{-2}$, $\sum_{l=1}^{t-k-1} l^{-3} < \sum_{l=1}^{\infty} l^{-3} < \infty$, and $\sum_{l=T-t+k+2}^{T} l^{-3} < \sum_{l=k+2}^{\infty} l^{-3} < \infty$ for $t = k+2, \ldots, T$. The above expression can be simplified if we set $h = i - j$ and $m = t - j$, and write that

$$\begin{aligned}
&\sum_{m=-\infty}^{\infty}\sum_{h=-\infty}^{\infty} |\gamma_\varepsilon(h)| I(0 \leq m \leq t-2) I(k \leq m-h \leq t-2) \\
&\times (C_1 t^{-2} + C_5 (T-t+1)^{-3} (t-1)^{-2})(C_3 (t-k)^{-2} + C_6 (T-t+k+1)^{-3}) \\
\leq &(t-1)(C_1 t^{-2} + C_5 (T-t+1)^{-3} (t-1)^{-2} + C_5 (T-t+1)^{-3} T^{-2}) \\
&\times (C_3 (t-k)^{-2} + C_6 (T-t+k+1)^{-3}) I(t \geq k+2)
\end{aligned}$$

$$\le (C_1 t^{-1} + C_5(T-t+1)^{-3}(t-1)^{-1} + C_5(T-t+1)^{-3}T^{-1})$$
$$\times (C_3(t-k)^{-2} + C_6(T-t+k+1)^{-3}) I(t \ge k+2),$$

where the first inequality follows from $\sum_{h=-\infty}^{\infty} |\gamma_\varepsilon(h)| < \infty$, and $\sum_{m=-\infty}^{\infty} I(0 \le m \le t-2) = t-1$. Therefore, we can conclude that

$$\lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} \sum_{j=1}^{T} \sum_{i=1}^{T} \gamma_\varepsilon(i-j) \sum_{s=1}^{j-1} \sum_{l=1}^{i-1} (\sum_{p=2}^{8} w_{Tts}^{p})(\sum_{q=2}^{8} w_{T,t-k,l}^{q}) I(2 \le j \le t) I(2 \le i \le t-k)$$
$$\le \lim_{T\to\infty} T^{-1} \sum_{t=k+2}^{T} (C_1 t^{-1} + C_5(T-t+1)^{-3}(t-1)^{-1} + C_5(T-t+1)^{-3}T^{-1})$$
$$\times (C_3(t-k)^{-2} + C_6(T-t+k+1)^{-3})$$
$$= 0.$$

□

Lemma 9. *Let*

$$S_{T,t}^{1} = \sum_{m=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} \gamma_\varepsilon(h) (\sum_{s=m+1}^{t-1} f_{T\lambda}(s)) (\sum_{l=m-h+1}^{t-1} f_{T\lambda}(l-k)) I(0 \le m \le t-2) I(k \le m-h \le t-2),$$

and

$$S^{1} = \sum_{m=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} \gamma_\varepsilon(h) (\sum_{s=m+1}^{\infty} f_{\lambda}(s)) (\sum_{l=m-h+1}^{\infty} f_{\lambda}(l-k)) I(0 \le m) I(k \le m-h),$$

where $\sum_{h=-\infty}^{\infty} |\gamma_\varepsilon(h)| < \infty$. *Then,*

$$\lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} S_{T,t}^{1} = S^{1}.$$

Proof of Lemma 9. We first show that $|\lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} S_{T,t}^{1}| < C$ where $C < \infty$ is a constant that does not depend on T, then we apply the dominated convergence theorem.

$$|T^{-1} \sum_{t=k+1}^{T} S_{T,t}^{1}|$$
$$\le T^{-1} \sum_{t=k+1}^{T} \sum_{m=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} |\gamma_\varepsilon(h)| \sum_{s=m+1}^{t-1} \sum_{l=m-h+1}^{t-1} |f_{T\lambda}(s)||f_{T\lambda}(l-k)|$$
$$\times I(0 \le m \le t-2) I(k \le m-h \le t-2)$$
$$\le C_1 T^{-1} \sum_{t=k+1}^{T} \sum_{m=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} |\gamma_\varepsilon(h)| \sum_{s=m+1}^{t-1} s^{-3} \sum_{l=m-h+1}^{t-1} (l-k)^{-3}$$
$$\times I(0 \le m \le t-2) I(k \le m-h \le t-2)$$
$$\le C_1 T^{-1} \sum_{t=k+1}^{T} \sum_{m=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} |\gamma_\varepsilon(h)| \sum_{s=m+1}^{\infty} s^{-3} \sum_{l=m-h+1}^{\infty} (l-k)^{-3} I(0 \le m) I(k \le m-h)$$

$$\leq C_2 \sum_{m=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} |\gamma_\varepsilon(h)|(m+1)^{-2}(m-h+1-k)^{-2} I(0 \leq m) I(k \leq m-h) < \infty,$$

where the second inequality follows from Property 2 in Appendix A. The fourth inequality is due to $(T-k)/T < 2$ for $k \in -T+1, \dots, 0, \dots T-1$. The last inequality is due to the assumption that $\sum_{h=-\infty}^{\infty} |\gamma_\varepsilon(h)| < \infty$.

Therefore, we can use the dominated convergence theorem to show that $\lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} S_{T,t}^1 = S^1$.

$$\lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} S_{T,t}^1$$

$$= \lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} \sum_{m=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} \sum_{s=m+1}^{\infty} \sum_{l=m-h+1}^{\infty} \gamma_\varepsilon(h) f_{T\lambda}(s) f_{T\lambda}(l-k)$$

$$\times I(0 \leq m \leq t-2) I(k \leq m-h \leq t-2) I(s \leq t-1) I(l \leq t-1)$$

$$= \sum_{m=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} \sum_{s=m+1}^{\infty} \sum_{l=m-h+1}^{\infty} \gamma_\varepsilon(h) I(0 \leq m) I(k \leq m-h)$$

$$\times \lim_{T\to\infty} f_{T\lambda}(s) f_{T\lambda}(l-k) T^{-1} \sum_{t=k+1}^{T} I(m \leq t-2) I(m-h \leq t-2) I(s \leq t-1) I(l \leq t-1)$$

$$= \sum_{m=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} \sum_{s=m+1}^{\infty} \sum_{l=m-h+1}^{\infty} \gamma_\varepsilon(h) f_\lambda(s) f_\lambda(l-k) I(0 \leq m) I(k \leq m-h)$$

$$\times \lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} I(m \leq t-2) I(m-h \leq t-2) I(s \leq t-1) I(l \leq t-1),$$

where the second equality is due to the dominated convergence theorem. Note that

$$\lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} I(m \leq t-2) I(m-h \leq t-2) I(s \leq t-1) I(l \leq t-1)$$

$$= \lim_{T\to\infty} T^{-1}(T - \max\{m+1, m-h+1, s, l, k\}) = 1.$$

□

Lemma 10. *Let* $\{\varepsilon_t\}_{t=-\infty}^{\infty}$ *be a weakly stationary process with* $\gamma_\varepsilon(i-j) = cov(\varepsilon_i, \varepsilon_j)$ *for* $i, j \in \mathbb{Z}$, *and* $\sum_{h=-\infty}^{\infty} |\gamma_\varepsilon(h)| < \infty$. *Then,*

$$\lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} \sum_{j=1}^{T} \sum_{i=1}^{T} \gamma_\varepsilon(i-j) \sum_{s=1}^{j-1} \sum_{l=i}^{T} w_{Tts} w_{T,t-k,l} I(2 \leq j \leq t) I(t-k+1 \leq i \leq T)$$

$$= \lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} \sum_{j=1}^{T} \sum_{i=1}^{T} \gamma_\varepsilon(i-j) \sum_{s=1}^{j-1} \sum_{l=i}^{T} f_{T\lambda}(t-s) f_{T\lambda}(t-k-l)$$

$$\times I(2 \leq j \leq t) I(t-k+1 \leq i \leq T).$$

Proof of Lemma 10. We use Property 1 to write that $w_{T,t-k,l} = w_{T,l,t-k}$ and the rest of the proof follows a similar argument as in the proof of Lemma 8 therefore we skip the discussion. □

Lemma 11. *Let*

$$S_{T,t}^2 = \sum_{m=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} \gamma_\varepsilon(h)(\sum_{s=m+1}^{t-1} f_{T\lambda}(s))(\sum_{l=t-T}^{m-h} f_{T\lambda}(l-k))$$
$$\times I(0 \le m \le t-2) I(t-T \le m-h \le k-1),$$

and

$$S^2 = \sum_{m=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} \gamma_\varepsilon(h)(\sum_{s=m+1}^{\infty} f_\lambda(s))(\sum_{l=-\infty}^{m-h} f_\lambda(l-k)) I(0 \le m) I(m-h \le k-1),$$

where $\sum_{h=-\infty}^{\infty} |\gamma_\varepsilon(h)| < \infty$*. Then,*

$$\lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} S_{T,t}^2 = S^2.$$

Proof of Lemma 11. We first show that $|\lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} S_{T,t}^2| < C$ where $C < \infty$ is a constant that does not depend on T, then we apply the dominated convergence theorem. The discussion, here, is the same as the discussion in the proof of Lemma 9; therefore, we will not give a complete proof. We will only establish that $\left|\lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} S_{T,t}^2\right|$ is bounded by a constant C that does not depend on T.

Note that

$$\begin{aligned}
&|T^{-1} \sum_{t=k+1}^{T} S_{T,t}^2| \\
\le& T^{-1} \sum_{t=k+1}^{T} \sum_{m=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} |\gamma_\varepsilon(h)| \sum_{s=m+1}^{t-1} |f_{T\lambda}(s)| \sum_{l=t-T}^{m-h} |f_{T\lambda}(l-k)| \\
&\times I(0 \le m \le t-2) I(t-T \le m-h \le k-1) \\
\le& C_1 T^{-1} \sum_{t=k+1}^{T} \sum_{m=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} |\gamma_\varepsilon(h)| \sum_{s=m+1}^{t-1} s^{-3} \sum_{l=t-T}^{m-h} |l-k|^{-3} \\
&\times I(0 \le m \le t-2) I(t-T \le m-h \le k-1) \\
\le& C_1 T^{-1} \sum_{t=k+1}^{T} \sum_{m=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} |\gamma_\varepsilon(h)| \sum_{s=m+1}^{\infty} s^{-3} \sum_{l=-\infty}^{m-h} |l-k|^{-3} I(0 \le m) I(m-h \le k-1) \\
\le& C_2 \sum_{m=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} |\gamma_\varepsilon(h)| (m+1)^{-2} (m-h-k)^{-2} I(0 \le m) I(m-h \le k-1) < \infty,
\end{aligned}$$

where the second inequality follows from Property 2 in Appendix A. The fourth inequality is due to $(T-k)/T < 2$ for $k = -T+1, \ldots, 0, \ldots, T-1$. The last inequality is because $\sum_{h=-\infty}^{\infty} |\gamma_\varepsilon(h)| < \infty$.

Therefore, we can make use of the dominated convergence theorem to show that $\lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} S_{T,t}^2 = S^2$. The rest of the proof is the same as the proof of Lemma 9, so we skip it.

□

Lemma 12. *Let $\{\varepsilon_t\}_{t=-\infty}^{\infty}$ be a weakly stationary process with $\gamma_\varepsilon(i-j) = cov(\varepsilon_i, \varepsilon_j)$ for $i, j \in \mathbb{Z}$, and $\sum_{h=-\infty}^{\infty} |\gamma_\varepsilon(h)| < \infty$. Then,*

$$\lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} \sum_{j=1}^{T} \sum_{i=1}^{T} \gamma_\varepsilon(i-j) \sum_{s=j}^{T} \sum_{l=1}^{i-1} w_{Tts} w_{T,t-k,l} I(t+1 \le j \le T) I(2 \le i \le t-k)$$
$$= \lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} \sum_{j=1}^{T} \sum_{i=1}^{T} \gamma_\varepsilon(i-j) \sum_{s=j}^{T} \sum_{l=1}^{i-1} f_{T\lambda}(t-s) f_{T\lambda}(t-k-l)$$
$$\times I(t+1 \le j \le T) I(2 \le i \le t-k).$$

Proof of Lemma 12. We use Property 1 to write that $w_{Tts} = w_{Tst}$ and the rest of the proof follows a similar argument as in the proof of Lemma 8 therefore we skip the discussion. □

Lemma 13. *Let*

$$S_{T,t}^3 = \sum_{m=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} \gamma_\varepsilon(h) (\sum_{s=t-T}^{m} f_{T\lambda}(s)) (\sum_{l=m-h+1}^{t-1} f_{T\lambda}(l-k))$$
$$\times I(t-T \le m \le -1) I(k \le m-h \le t-2),$$

and

$$S^3 = \sum_{m=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} \gamma_\varepsilon(h) (\sum_{s=-\infty}^{m} f_\lambda(s)) (\sum_{l=m-h+1}^{\infty} f_\lambda(l-k)) I(m \le -1) I(k \le m-h),$$

where $\sum_{h=-\infty}^{\infty} |\gamma_\varepsilon(h)| < \infty$. Then,

$$\lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} S_{T,t}^3 = S^3.$$

Proof of Lemma 13. The proof of this lemma is analogous to the proof of Lemma 11; therefore, we skip the discussion. □

Lemma 14. *Let $\{\varepsilon_t\}_{t=-\infty}^{\infty}$ be a weakly stationary process with $\gamma_\varepsilon(i-j) = cov(\varepsilon_i, \varepsilon_j)$ for $i, j \in \mathbb{Z}$, and $\sum_{h=-\infty}^{\infty} |\gamma_\varepsilon(h)| < \infty$. Then,*

$$\lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} \sum_{j=1}^{T} \sum_{i=1}^{T} \gamma_\varepsilon(i-j) \sum_{s=j}^{T} \sum_{l=i}^{T} w_{Tts} w_{T,t-k,l} I(t+1 \le j \le T) I(t+1 \le i \le T)$$
$$= \lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} \sum_{j=1}^{T} \sum_{i=1}^{T} \gamma_\varepsilon(i-j) \sum_{s=j}^{T} \sum_{l=i}^{T} f_{T\lambda}(t-s) f_{T\lambda}(t-k-l)$$
$$\times I(t+1 \le j \le T) I(t+1 \le i \le T).$$

Proof of Lemma 14. We use Property 1 in Appendix A to write that $w_{Tts} w_{T,t-k,l} = w_{Tst} w_{T,l,t-k}$ and the rest of the proof follows a similar argument as in the proof of Lemma 8 therefore we skip the discussion. □

Lemma 15. *Let*

$$S^4_{T,t} = \sum_{m=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} \gamma_\varepsilon(h)(\sum_{s=t-T}^{m} f_{T\lambda}(s))(\sum_{l=t-T}^{m-h} f_{T\lambda}(l-k))$$
$$\times I(t-T \leq m \leq -1)I(t-T \leq m-h \leq k-1),$$

and

$$S^4 = \sum_{m=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} \gamma_\varepsilon(h)(\sum_{s=-\infty}^{m} f_\lambda(s))(\sum_{l=-\infty}^{m-h} f_\lambda(l-k))I(m \leq -1)I(m-h \leq k-1).$$

where $\sum_{h=-\infty}^{\infty} |\gamma_\varepsilon(h)| < \infty$. *Then,*

$$\lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} S^4_{T,t} = S^4.$$

Proof of Lemma 15. The proof of this lemma is analogous to the proof of Lemma 9 therefore we skip the discussion. □

Proof of Theorem 1. Note that $\hat{c}_{Tt}$ can be expressed as[3]

$$\hat{c}_{Tt} = u_t - \sum_{s=1}^{T} w_{Tts}u_s - \alpha_3 \sum_{j=1}^{T} \varepsilon_j (\sum_{s=j}^{T} w_{Tts} - I(j \leq t)).$$

Note that $u_t - \sum_{s=1}^{T} w_{Tts}u_s$ is the cyclical component of u_t. The autocovariance function of $\hat{c}_{Tt}$ is

$$\lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} cov(\hat{c}_{Tt}, \hat{c}_{T,t-k})$$
$$= \lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} cov(u_t - \sum_{s=1}^{T} w_{Tts}u_s - \alpha_3 \sum_{j=1}^{T} \varepsilon_j(\sum_{s=j}^{T} w_{Tts} - I(j \leq t))$$
$$, u_{t-k} - \sum_{l=1}^{T} w_{T,t-k,l}u_l - \alpha_3 \sum_{i=1}^{T} \varepsilon_i(\sum_{l=i}^{T} w_{T,t-k,l} - I(i \leq t-k)))$$
$$= \lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} cov(u_t - \sum_{s=1}^{T} w_{Tts}u_s, u_{t-k} - \sum_{l=1}^{T} w_{T,t-k,l}u_l) \tag{17}$$
$$- \alpha_3 \lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} cov(u_t - \sum_{s=1}^{T} w_{Tts}u_s, \sum_{i=1}^{T} \varepsilon_i(\sum_{l=i}^{T} w_{T,t-k,l} - I(i \leq t-k))) \tag{18}$$
$$- \alpha_3 \lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} cov(\sum_{j=1}^{T} \varepsilon_j(\sum_{s=j}^{T} w_{Tts} - I(j \leq t)), u_{t-k} - \sum_{l=1}^{T} w_{T,t-k,l}u_l) \tag{19}$$
$$+ \alpha_3^2 \lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} cov(\sum_{j=1}^{T} \varepsilon_j(\sum_{s=j}^{T} w_{Tts} - I(j \leq t)), \sum_{i=1}^{T} \varepsilon_i(\sum_{l=i}^{T} w_{T,t-k,l} - I(i \leq t-k))). \tag{20}$$

Note that the expression in Equation (17) is the autocovariance function of the cyclical component of a weakly stationary process. Therefore, Lemma 5 implies that

$$\lim_{T\to\infty} T^{-1} \sum_{t=k+1}^{T} cov(u_t - \sum_{s=1}^{T} w_{Tts}u_s, u_{t-k} - \sum_{l=1}^{T} w_{T,t-k,l}u_l)$$

$$=\gamma_u(k) - 2\sum_{j=-\infty}^{\infty}\gamma_u(j)f_\lambda(j+k) + \sum_{h=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\gamma_u(h)f_\lambda(h+j)f_\lambda(k+j).$$

Furthermore, the expressions in Equations (18) and (19) are zero since $cov(u_t, \varepsilon_{t-k}) = 0$ for any $k \in \mathbb{Z}$. Thus, we only need to consider the expression in Equation (20).

$$\begin{aligned}
&\alpha_3^2 \lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T} cov(\sum_{j=1}^{T}\varepsilon_j(\sum_{s=j}^{T} w_{Tts} - I(j\le t)), \sum_{i=1}^{T}\varepsilon_i(\sum_{l=i}^{T} w_{T,t-k,l} - I(i \le t-k)))\\
=&\alpha_3^2 \lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T}\sum_{j=1}^{T}\sum_{i=1}^{T} cov(\varepsilon_j,\varepsilon_i)(\sum_{s=j}^{T} w_{Tts} - I(j\le t))(\sum_{l=i}^{T} w_{T,t-k,l} - I(i\le t-k))\\
=&\alpha_3^2 \lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T}\sum_{j=1}^{T}\sum_{i=1}^{T}\gamma_\varepsilon(i-j)(\sum_{s=1}^{j-1} w_{Tts})(\sum_{l=1}^{i-1} w_{T,t-k,l}) \qquad (21)\\
&\times I(2\le j\le t)I(2\le i\le t-k)\\
&-\alpha_3^2 \lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T}\sum_{j=1}^{T}\sum_{i=1}^{T}\gamma_\varepsilon(i-j)(\sum_{s=1}^{j-1} w_{Tts})(\sum_{l=i}^{T} w_{T,t-k,l}) \qquad (22)\\
&\times I(2\le j\le t)I(t-k+1\le i\le T)\\
&-\alpha_3^2 \lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T}\sum_{j=1}^{T}\sum_{i=1}^{T}\gamma_\varepsilon(i-j)(\sum_{s=j}^{T} w_{Tts})(\sum_{l=1}^{i-1} w_{T,t-k,l}) \qquad (23)\\
&\times I(t+1\le j\le T)I(2\le i\le t-k)\\
&+\alpha_3^2 \lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T}\sum_{j=1}^{T}\sum_{i=1}^{T}\gamma_\varepsilon(i-j)(\sum_{s=j}^{T} w_{Tts})(\sum_{l=i}^{T} w_{T,t-k,l}) \qquad (24)\\
&\times I(t+1\le j\le T)I(t-k+1\le i\le T).
\end{aligned}$$

The last equality holds because $\sum_{s=j}^{T} w_{Tts} - I(j\le t) = \sum_{s=1}^{j-1} w_{Tts} I(2\le j\le t) + \sum_{s=j}^{T} w_{Tts} I(t+1 \le j \le T)$ and $\sum_{s=1}^{T} w_{Tts} = 1$. A similar manipulation applies for $\sum_{l=i}^{T} w_{T,t-k,l} - I(i\le t-k)$, as well.

Now consider the expression in Equation (21) and note that

$$\begin{aligned}
&\lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T}\sum_{j=1}^{T}\sum_{i=1}^{T}\gamma_\varepsilon(i-j)(\sum_{s=1}^{j-1} w_{Tts})(\sum_{l=1}^{i-1} w_{T,t-k,l}) I(2\le j\le t)I(2\le i\le t-k)\\
=&\lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T}\sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty}\gamma_\varepsilon(i-j)(\sum_{s=1}^{j-1} f_{T\lambda}(t-s))(\sum_{l=1}^{i-1} f_{T\lambda}(t-l-k))\\
&\times I(2\le j\le t)I(2\le i\le t-k)\\
=&\lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T}\sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty}\gamma_\varepsilon(i-j)(\sum_{s=t-j+1}^{t-1} f_{T\lambda}(s))(\sum_{l=t-i+1}^{t-1} f_{T\lambda}(l-k))\\
&\times I(2\le j\le t)I(2\le i\le t-k),
\end{aligned}$$

where the first equality follows from Lemma 8. Next, we set $h = i - j$ and $m = t - j$, and write that

$$\lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T}\sum_{m=-\infty}^{\infty}\sum_{h=-\infty}^{\infty}\gamma_\varepsilon(h)(\sum_{s=m+1}^{t-1} f_{T\lambda}(s))(\sum_{l=m-h+1}^{t-1} f_{T\lambda}(l-k))$$

$$\times I(0\le m\le t-2)I(k\le m-h\le t-2)$$

$$=\sum_{m=-\infty}^{\infty}\sum_{h=-\infty}^{\infty}\gamma_\varepsilon(h)(\sum_{s=m+1}^{\infty} f_\lambda(s))(\sum_{l=m-h+1}^{\infty} f_\lambda(l-k))I(0\le m)I(k\le m-h)$$

$$=\sum_{m=-\infty}^{\infty}\sum_{h=-\infty}^{\infty}\gamma_\varepsilon(h-k)(\sum_{s=m+1}^{\infty} f_\lambda(s))(\sum_{l=m-h+k+1}^{\infty} f_\lambda(l-k))I(0\le m)I(k\le m-h+k)$$

$$=\sum_{m=-\infty}^{\infty}\sum_{h=-\infty}^{\infty}\gamma_\varepsilon(h-k)(\sum_{s=m+1}^{\infty} f_\lambda(s))(\sum_{l=m+1}^{\infty} f_\lambda(l-h))I(0\le m)I(h\le m),$$

where the first equality follows from Lemma 9 and the last two equalities are the result of the change of variables h and l.

Next, consider the expression in Equation (22) and note that

$$\lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T}\sum_{j=1}^{T}\sum_{i=1}^{T}\gamma_\varepsilon(i-j)(\sum_{s=1}^{j-1} w_{Tts})(\sum_{l=i}^{T} w_{T,t-k,l})I(2\le j\le t)I(t-k+1\le i\le T)$$

$$=\lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T}\sum_{j=1}^{T}\sum_{i=1}^{T}\gamma_\varepsilon(i-j)(\sum_{s=1}^{j-1} f_{T\lambda}(t-s))(\sum_{l=i}^{T} f_{T\lambda}(t-k-l))$$

$$\times I(2\le j\le t)I(t-k+1\le i\le T)$$

$$=\lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T}\sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty}\gamma_\varepsilon(i-j)(\sum_{s=t-j+1}^{t-1} f_{T\lambda}(s))(\sum_{l=t-T}^{t-i} f_{T\lambda}(l-k))$$

$$\times I(0\le t-j\le t-2)I(t-T\le t-i\le k-1),$$

where the first equality follows from Lemma 10. Next, we set $h=i-j$ and $m=t-j$, and write that

$$\lim_{T\to\infty} T^{-1}\sum_{t=k+1}^{T}\sum_{m=-\infty}^{\infty}\sum_{h=-\infty}^{\infty}\gamma_\varepsilon(h)(\sum_{s=m+1}^{t-1} f_{T\lambda}(s))(\sum_{l=t-T}^{m-h} f_{T\lambda}(l-k))$$

$$\times I(0\le m\le t-2)I(t-T\le m-h\le k-1)$$

$$=\sum_{m=-\infty}^{\infty}\sum_{h=-\infty}^{\infty}\gamma_\varepsilon(h)(\sum_{s=m+1}^{\infty} f_\lambda(s))(\sum_{l=-\infty}^{m-h} f_\lambda(l-k))I(0\le m)I(m-h\le k-1)$$

$$=\sum_{m=-\infty}^{\infty}\sum_{h=-\infty}^{\infty}\gamma_\varepsilon(h-k)(\sum_{s=m+1}^{\infty} f_\lambda(s))(\sum_{l=-\infty}^{m-h+k} f_\lambda(l-k))I(0\le m)I(m-h+k\le k-1)$$

$$=\sum_{m=-\infty}^{\infty}\sum_{h=-\infty}^{\infty}\gamma_\varepsilon(h-k)(\sum_{s=m+1}^{\infty} f_\lambda(s))(\sum_{l=-\infty}^{m} f_\lambda(l-h))I(0\le m)I(m+1\le h),$$

the first equality follows from Lemma 11 and the last two equalities are the result of the change of variables h and l.

The arguments for the expressions in Equations (23) and (24) are similar to the above ones. The result for Equation (23) is obtained by Lemmas 12 and 13. The result for Equation (24) is obtained by Lemmas 14 and 15. We skip the discussion for the sake of brevity. □

Proof of Theorem 2. First, we will show that the spectrum of the cyclical component exists by showing that $\sum_{k=-\infty}^{\infty}|\gamma_c(k)|<\infty$.

$$
\begin{aligned}
&\sum_{k=-\infty}^{\infty}|\gamma_c(k)| \\
\leq &\sum_{k=-\infty}^{\infty}|\gamma_u(k)| + 2\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}|\gamma_u(j)||f_\lambda(j+k)| &(25)\\
&+\sum_{k=-\infty}^{\infty}\sum_{h=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}|\gamma_u(h)||f_\lambda(h+j)||f_\lambda(k+j)| &(26)\\
&+\alpha_3^2\sum_{k=-\infty}^{\infty}\sum_{m=0}^{\infty}\sum_{h=-\infty}^{m}|\gamma_\varepsilon(h-k)|\sum_{s=m+1}^{\infty}\sum_{l=m+1}^{\infty}|f_\lambda(s)||f_\lambda(l-h)| &(27)\\
&+\alpha_3^2\sum_{m=0}^{\infty}\sum_{h=m+1}^{\infty}|\gamma_\varepsilon(h-k)|\sum_{s=m+1}^{\infty}\sum_{l=-\infty}^{m}|f_\lambda(s)||f_\lambda(l-h)| &(28)\\
&+\alpha_3^2\sum_{m=-\infty}^{-1}\sum_{h=-\infty}^{m}|\gamma_\varepsilon(h-k)|\sum_{s=-\infty}^{m}\sum_{l=m+1}^{\infty}|f_\lambda(s)||f_\lambda(l-h)| &(29)\\
&+\alpha_3^2\sum_{m=-\infty}^{-1}\sum_{h=m+1}^{\infty}|\gamma_\varepsilon(h-k)|\sum_{s=-\infty}^{m}\sum_{l=-\infty}^{m}|f_\lambda(s)||f_\lambda(l-h)|, &(30)
\end{aligned}
$$

note that we have shown in Lemma 7 that the sum of the expressions in Equations (25) and (26) is finite.

When we consider the expression in Equation (27), it is easy to see that

$$
\begin{aligned}
&\sum_{k=-\infty}^{\infty}\sum_{m=0}^{\infty}\sum_{h=-\infty}^{m}|\gamma_\varepsilon(h-k)|\sum_{s=m+1}^{\infty}\sum_{l=m+1}^{\infty}|f_\lambda(s)||f_\lambda(l-h)| &(31)\\
\leq &C\sum_{k=-\infty}^{\infty}\sum_{m=0}^{\infty}\sum_{h=-\infty}^{m}|\gamma_\varepsilon(h-k)|(m+1)^{-2}(m+1-h)^{-2}\\
= &C\sum_{k=-\infty}^{\infty}\sum_{m=0}^{\infty}\sum_{h=-\infty}^{m}|\gamma_\varepsilon(k)|(m+1)^{-2}(m+1-h)^{-2}<\infty,
\end{aligned}
$$

by $\sum_{k=-\infty}^{\infty}|\gamma_\varepsilon(k)|<\infty$. It can be shown that the expressions in Equations (28)-(30) are finite by following a similar argument as above. We skip the discussion and conclude that $\sum_{k=-\infty}^{\infty}|\gamma_c(k)|<\infty$ and the spectrum of $\hat{c}_{Tt}$ exists.

Next, we want to derive the spectrum of $\hat{c}_{Tt}$ by writing that

$$
\begin{aligned}
I_c(\omega) = &(2\pi)^{-1}\sum_{k=-\infty}^{\infty}(\gamma_u(k) - 2\sum_{j=-\infty}^{\infty}\gamma_u(j)f_\lambda(j+k) &(32)\\
&+\sum_{h=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\gamma_u(h)f_\lambda(h+j)f_\lambda(k+j))e^{-i\omega k} &(33)
\end{aligned}
$$

$$+\alpha_3^2(2\pi)^{-1}\sum_{k=-\infty}^{\infty}(\sum_{m=0}^{\infty}\sum_{h=-\infty}^{m}\gamma_\varepsilon(h-k)(\sum_{s=m+1}^{\infty}f_\lambda(s))(\sum_{l=m+1}^{\infty}f_\lambda(l-h)))e^{-i\omega k} \tag{34}$$

$$-\alpha_3^2(2\pi)^{-1}\sum_{k=-\infty}^{\infty}(\sum_{m=0}^{\infty}\sum_{h=m+1}^{\infty}\gamma_\varepsilon(h-k)(\sum_{s=m+1}^{\infty}f_\lambda(s))(\sum_{l=-\infty}^{m}f_\lambda(l-h)))e^{-i\omega k} \tag{35}$$

$$-\alpha_3^2(2\pi)^{-1}\sum_{k=-\infty}^{\infty}(\sum_{m=-\infty}^{-1}\sum_{h=-\infty}^{m}\gamma_\varepsilon(h-k)(\sum_{s=-\infty}^{m}f_\lambda(s))(\sum_{l=m+1}^{\infty}f_\lambda(l-h)))e^{-i\omega k} \tag{36}$$

$$+\alpha_3^2(2\pi)^{-1}\sum_{k=-\infty}^{\infty}(\sum_{m=-\infty}^{-1}\sum_{h=m+1}^{\infty}\gamma_\varepsilon(h-k)(\sum_{s=-\infty}^{m}f_\lambda(s))(\sum_{l=-\infty}^{m}f_\lambda(l-h)))e^{-i\omega k}, \tag{37}$$

by plugging in $\gamma_c(k)$ that was obtained in Theorem 1.

Note that the sum of the expressions in Equations (32) and (33) was already derived in Lemma 7; therefore, it is equal to $(1-h_\lambda(\omega))^2 I_u(\omega)$.

Next, consider the expression in Equation (34) and note that

$$\begin{aligned}
&(2\pi)^{-1}\sum_{k=-\infty}^{\infty}(\sum_{m=0}^{\infty}\sum_{h=-\infty}^{m}\gamma_\varepsilon(h-k)(\sum_{s=m+1}^{\infty}f_\lambda(s))(\sum_{l=m+1}^{\infty}f_\lambda(l-h)))e^{-i\omega k}\\
=&(2\pi)^{-1}\sum_{m=0}^{\infty}\sum_{h=-\infty}^{m}(\sum_{s=m+1}^{\infty}f_\lambda(s))(\sum_{l=m+1}^{\infty}f_\lambda(l-h))\sum_{k=-\infty}^{\infty}\gamma_\varepsilon(h-k)e^{-i\omega k},
\end{aligned}$$

the infinite sums can be interchanged since Fubini's Theorem follows by the argument in Equation (31). We continue our argument by writing that

$$\begin{aligned}
&(2\pi)^{-1}\sum_{m=0}^{\infty}\sum_{h=-\infty}^{m}(\sum_{s=m+1}^{\infty}f_\lambda(s))(\sum_{l=m+1}^{\infty}f_\lambda(l-h))\sum_{k=-\infty}^{\infty}\gamma_\varepsilon(h-k)e^{-i\omega k},\\
=&(2\pi)^{-1}\sum_{m=0}^{\infty}\sum_{h=-\infty}^{m}(\sum_{s=m+1}^{\infty}f_\lambda(s))(\sum_{l=m+1}^{\infty}f_\lambda(l-h))e^{-i\omega h}\sum_{k=-\infty}^{\infty}\gamma_\varepsilon(k)e^{i\omega k}\\
=&I_\varepsilon(\omega)\sum_{m=0}^{\infty}\sum_{s=m+1}^{\infty}f_\lambda(s)\sum_{h=-\infty}^{m}\sum_{l=m+1}^{\infty}f_\lambda(l-h)e^{-i\omega h}\\
=&I_\varepsilon(\omega)\sum_{m=0}^{\infty}\sum_{s=m+1}^{\infty}f_\lambda(s)\sum_{l=0}^{\infty}\sum_{h=-\infty}^{m}f_\lambda(l+m+1-h)e^{-i\omega h}\\
=&I_\varepsilon(\omega)\sum_{m=0}^{\infty}\sum_{s=m+1}^{\infty}f_\lambda(s)\sum_{l=0}^{\infty}\sum_{h=l+1}^{\infty}f_\lambda(h)e^{-i\omega(l+m+1-h)}\\
=&I_\varepsilon(\omega)\sum_{m=0}^{\infty}\sum_{s=m+1}^{\infty}f_\lambda(s)e^{-i\omega(m+1)}\sum_{l=0}^{\infty}\sum_{h=l+1}^{\infty}f_\lambda(h)e^{-i\omega(l-h)}\\
=&\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}f_\lambda(s+m+1)e^{-i\omega(m+1)}\sum_{l=0}^{\infty}\sum_{h=l+1}^{\infty}f_\lambda(h)e^{i\omega(h-l)}\\
=&I_\varepsilon(\omega)\sum_{s=0}^{\infty}\sum_{m=s+1}^{\infty}f_\lambda(m)e^{-i\omega(m-s)}\sum_{l=0}^{\infty}\sum_{h=l+1}^{\infty}f_\lambda(h)e^{i\omega(h-l)}.
\end{aligned} \tag{38}$$

It can be shown that the expressions in Equations (35)-(37) are equal to

$$I_\varepsilon(\omega)\sum_{s=0}^{\infty}\sum_{m=s+1}^{\infty}f_\lambda(m)e^{-i\omega(m-s-1)}\sum_{l=0}^{\infty}\sum_{h=l+1}^{\infty}f_\lambda(h)e^{-i\omega(h-l)}, \tag{39}$$

$$I_\varepsilon(\omega)\sum_{s=0}^{\infty}\sum_{m=s+1}^{\infty} f_\lambda(m)e^{i\omega(m-s-1)}\sum_{l=0}^{\infty}\sum_{h=l+1}^{\infty} f_\lambda(h)e^{i\omega(h-l)}, \tag{40}$$

$$I_\varepsilon(\omega)\sum_{s=0}^{\infty}\sum_{m=s+1}^{\infty} f_\lambda(m)e^{i\omega(m-s)}\sum_{l=0}^{\infty}\sum_{h=l+1}^{\infty} f_\lambda(h)e^{-i\omega(h-l)}, \tag{41}$$

respectively by interchanging and reindexing summations as in the case of Equation (34) above.

When we combine the expressions in the equations (38)-(41), we have

$$\begin{aligned}
&I_\varepsilon(\omega)\sum_{s=0}^{\infty}\sum_{m=s+1}^{\infty} f_\lambda(m)e^{-i\omega(m-s)}\sum_{l=0}^{\infty}\sum_{h=l+1}^{\infty} f_\lambda(h)e^{i\omega(h-l)}\\
&\quad - I_\varepsilon(\omega)\sum_{s=0}^{\infty}\sum_{m=s+1}^{\infty} f_\lambda(m)e^{-i\omega(m-s-1)}\sum_{l=0}^{\infty}\sum_{h=l+1}^{\infty} f_\lambda(h)e^{-i\omega(h-l)}\\
&\quad - I_\varepsilon(\omega)\sum_{s=0}^{\infty}\sum_{m=s+1}^{\infty} f_\lambda(m)e^{i\omega(m-s-1)}\sum_{l=0}^{\infty}\sum_{h=l+1}^{\infty} f_\lambda(h)e^{i\omega(h-l)}\\
&\quad + I_\varepsilon(\omega)\sum_{s=0}^{\infty}\sum_{m=s+1}^{\infty} f_\lambda(m)e^{i\omega(m-s)}\sum_{l=0}^{\infty}\sum_{h=l+1}^{\infty} f_\lambda(h)e^{-i\omega(h-l)}\\
=&I_\varepsilon(\omega)\sum_{s=0}^{\infty}\sum_{m=s+1}^{\infty}\sum_{l=0}^{\infty}\sum_{h=l+1}^{\infty} f_\lambda(m)f_\lambda(h)(e^{-i\omega(m-s-h+l)}+e^{i\omega(m-s-h+l)})\\
&\quad - I_\varepsilon(\omega)\sum_{s=0}^{\infty}\sum_{m=s+1}^{\infty}\sum_{l=0}^{\infty}\sum_{h=l+1}^{\infty} f_\lambda(m)f_\lambda(h)(e^{-i\omega(m-s-1+h-l)}+e^{i\omega(m-s-1+h-l)})\\
=&2I_\varepsilon(\omega)\sum_{s=0}^{\infty}\sum_{m=s+1}^{\infty}\sum_{l=0}^{\infty}\sum_{h=l+1}^{\infty} f_\lambda(m)f_\lambda(h)(\cos(\omega(m-s-h+l))-\cos(\omega(m-s-1+h-l)))\\
=&4I_\varepsilon(\omega)\sum_{s=0}^{\infty}\sum_{m=s+1}^{\infty}\sum_{l=0}^{\infty}\sum_{h=l+1}^{\infty} f_\lambda(m)f_\lambda(h)\sin(\omega(m-s-1/2))\sin(\omega(h-l-1/2)) \qquad (42)\\
=&I_\varepsilon(\omega)(2\sum_{s=0}^{\infty}\sum_{m=s+1}^{\infty} f_\lambda(m)\sin(\omega(m-s-1/2)))^2, \qquad (43)
\end{aligned}$$

where the second equality follows from Euler's formula and the third equality follows from the trigonometric identity; $\cos(\alpha)-\cos(\beta)=2\sin((\alpha+\beta)/2)\sin((\beta-\alpha)/2)$ when we set $\alpha=\omega(m-s-h+l)$ and $\beta=\omega(m-s-1+h-l)$.

We can simplify the expression in Equation (43) by writing that

$$2\sum_{s=0}^{\infty}\sum_{m=s+1}^{\infty} f_\lambda(m)\sin(\omega(m-s-1/2)) = 2\sum_{m=1}^{\infty} f_\lambda(m)\sum_{s=0}^{m-1}\sin(\omega(m-s-1/2)). \tag{44}$$

Using the identity $\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\sin(\beta)\cos(\alpha)$ by setting $\alpha=(m-1/2)\omega$ and $\beta=s\omega$ gives that

$$\begin{aligned}
&\sum_{s=0}^{m-1}\sin((m-s-1/2)\omega)\\
=&\sin((m-1/2)\omega)\sum_{s=0}^{m-1}\cos(s\omega)-\cos((m-1/2)\omega)\sum_{s=0}^{m-1}\sin(s\omega)
\end{aligned}$$

$$=\sin((m-1/2)\omega)(\frac{1}{2}+\frac{\sin((m-1/2)\omega)}{2\sin(\omega/2)})-\cos((m-1/2)\omega)(\frac{\cos(\omega/2)}{2\sin(\omega/2)}-\frac{\cos((m-1/2)\omega)}{2\sin(\omega/2)}),$$

where the last equality follows from two identities[4]. We can rewrite the above expression as

$$\begin{aligned}&(2\sin(\omega/2))^{-1}\sin((m-1/2)\omega)(\sin(\omega/2)+\sin((m-1/2)\omega))\\&-(2\sin(\omega/2))^{-1}\cos((m-1/2)\omega)(\cos(\omega/2)-\cos((m-1/2)\omega))\\=&(2\sin(\omega/2))^{-1}(\cos((m-1/2)\omega)^2+\sin((m-1/2)\omega)^2)\\&-(2\sin(\omega/2))^{-1}(\cos((m-1/2)\omega)\cos(\omega/2)-\sin((m-1/2)\omega)\sin(\omega/2))\\=&(2\sin(\omega/2))^{-1}(1-\cos(m\omega)),\end{aligned}$$

since $\cos((m-1/2)\omega)^2+\sin((m-1/2)\omega)^2=1$ and $\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$ by setting $\alpha=(m-1/2)\omega$ and $\beta=\omega/2$. Therefore, Equation (44) is equal to

$$\begin{aligned}&(2\sin(\omega/2))^{-1}2\sum_{m=1}^{\infty}(1-\cos(m\omega))f_\lambda(m)\\=&(2\sin(\omega/2))^{-1}\sum_{m=-\infty}^{\infty}(1-\cos(m\omega))f_\lambda(m)\\=&(2\sin(\omega/2))^{-1}(1-h_\lambda(\omega)),\end{aligned}$$

since $f_\lambda(m)=f_\lambda(-m)$ by Property 5, $\sum_{m=-\infty}^{\infty}f_\lambda(m)=1$ by Property 6, and $\sum_{m=-\infty}^{\infty}\cos(m\omega)f_\lambda(m)=h_\lambda(\omega)$ by Lemma 6.

The above manipulation simplifies our result in Equation (43) to $(1-h_\lambda(\omega))^2(2\sin(\omega/2))^{-2}I_\varepsilon(\omega)$. Therefore, we conclude that $I_c(\omega)=(1-h_\lambda(\omega))^2I_u(\omega)+\alpha_3^2(1-h_\lambda(\omega))^2(2\sin(\omega/2))^{-2}I_\varepsilon(\omega)$. □

Proof of Theorem 3. We write that

$$\begin{aligned}&\lim_{T\to\infty}T^{-1}\sum_{t=k+1}^{T}cov(\hat{c}^1_{Tt},\hat{c}^2_{T,t-k})\\=&\lim_{T\to\infty}T^{-1}\sum_{t=k+1}^{T}cov(\sum_{s=1}^{T}(I(t=s)-w_{Tts}(\lambda_1))y_{1s},\sum_{l=1}^{T}(I(t-k=l)-w_{T,t-k,l}(\lambda_2))y_{2l}),\end{aligned}$$

by using the definition of the cyclical component. Note that the series y_{1t} and y_{2t} are filtered with different values of the smoothing parameters (i.e. λ_1 and λ_2). The above expression can be written as

$$\begin{aligned}&\lim_{T\to\infty}T^{-1}\sum_{t=k+1}^{T}cov(\sum_{s=1}^{T}(I(t=s)-w_{Tts}(\lambda_1))(\phi y_{2s}+u_s),\sum_{l=1}^{T}(I(t-k=l)-w_{T,t-k,l}(\lambda_2))y_{2l})\\=&\lim_{T\to\infty}\phi T^{-1}\sum_{t=k+1}^{T}cov(\sum_{s=1}^{T}(I(t=s)-w_{Tts}(\lambda_1))y_{2s},\sum_{l=1}^{T}(I(t-k=l)-w_{T,t-k,l}(\lambda_2))y_{2l}),\quad(45)\end{aligned}$$

by using the cointegration identity between y_{1t} and y_{2t}. The last equality follows from the fact that $cov(u_s,y_{2l})=0$ for any $s,l\in\mathbb{Z}$. Note that

$$\sum_{s=1}^{T}(I(t=s)-w_{Tts}(\lambda_1))y_{2s}=-\sum_{j=1}^{T}\varepsilon_j(\sum_{s=j}^{T}w_{Tts}(\lambda_1)-I(j\le t)),$$

and

$$\sum_{l=1}^{T}(I(t-k=l)-w_{T,t-k,l}(\lambda_2))y_{2l} = -\sum_{i=1}^{T}\varepsilon_i(\sum_{l=i}^{T}w_{T,t-k,l}(\lambda_2)-I(i\le t-k)).$$

When we plug these two identities in the expression in Equation (45), we obtain

$$\lim_{T\to\infty}\phi T^{-1}\sum_{t=k+1}^{T}cov(\sum_{j=1}^{T}\varepsilon_j(\sum_{s=j}^{T}w_{Tts}(\lambda_1)-I(j\le t)),\sum_{i=1}^{T}\varepsilon_i(\sum_{l=i}^{T}w_{T,t-k,l}(\lambda_2)-I(i\le t-k))).$$

The result for the above expression was derived in the proof of Theorem 1 where we assumed that $\lambda_1=\lambda_2$. □

Proof of Theorem 4. We use the result of Theorem 3, and write that

$$\begin{aligned}
&I_c^{1,2}(\omega)=(2\pi)^{-1}\sum_{k=-\infty}^{\infty}\gamma_c^{1,2}(k)e^{-i\omega k}\\
=&\phi(2\pi)^{-1}\sum_{k=-\infty}^{\infty}\sum_{m=0}^{\infty}\sum_{h=-\infty}^{m}\gamma_\varepsilon(h-k)(\sum_{s=m+1}^{\infty}f_{\lambda_1}(s))(\sum_{l=m+1}^{\infty}f_{\lambda_2}(l-h))e^{-i\omega k}\\
&-\phi(2\pi)^{-1}\sum_{k=-\infty}^{\infty}\sum_{m=0}^{\infty}\sum_{h=m+1}^{\infty}\gamma_\varepsilon(h-k)(\sum_{s=m+1}^{\infty}f_{\lambda_1}(s))(\sum_{l=-\infty}^{m}f_{\lambda_2}(l-h))e^{-i\omega k}\\
&-\phi(2\pi)^{-1}\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{-1}\sum_{h=-\infty}^{m}\gamma_\varepsilon(h-k)(\sum_{s=-\infty}^{m}f_{\lambda_1}(s))(\sum_{l=m+1}^{\infty}f_{\lambda_2}(l-h))e^{-i\omega k}\\
&+\phi(2\pi)^{-1}\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{-1}\sum_{h=m+1}^{\infty}\gamma_\varepsilon(h-k)(\sum_{s=-\infty}^{m}f_{\lambda_1}(s))(\sum_{l=-\infty}^{m}f_{\lambda_2}(l-h))e^{-i\omega k}\\
=&4\phi I_\varepsilon(\omega)\sum_{s=0}^{\infty}\sum_{m=s+1}^{\infty}\sum_{l=0}^{\infty}\sum_{h=l+1}^{\infty}f_{\lambda_1}(m)f_{\lambda_2}(h)\sin(\omega(m-s-1/2))\sin(\omega(h-l-1/2)),
\end{aligned}$$

by using the derivation of the expression in Equation (42) in the proof of Theorem 2. It was also shown in the proof of Theorem 2 that

$$\sum_{s=0}^{\infty}\sum_{m=s+1}^{\infty}f_{\lambda_1}(m)\sin(\omega(m-s-1/2))=(1-h_{\lambda_1}(\omega))(2\sin(\omega/2))^{-1},$$

and

$$\sum_{l=0}^{\infty}\sum_{h=l+1}^{\infty}f_{\lambda_2}(h)\sin(\omega(h-l-1/2))=(1-h_{\lambda_2}(\omega))(2\sin(\omega/2))^{-1}.$$

Therefore, we can conclude that

$$I_c^{1,2}(\omega)=\phi(1-h_{\lambda_1}(\omega))(1-h_{\lambda_2}(\omega))(2\sin(\omega/2))^{-2}I_\varepsilon(\omega).$$

□

Notes

[1]Euler's formula: $e^{-i\omega k} = \cos(\omega k) - i\sin(\omega k)$.

Note that $\cos(-\omega k) = \cos(\omega k)$, and $\sin(-\omega k) = -\sin(\omega k)$

[2]For the details of uniform convergence of Fourier series please see Brown and Churchill (2001), Chapter 2, Section 22.

[3]see Theorem 5 of de Jong and Sakarya (2016) for the details.

[4]$\sum_{s=0}^{m-1}\cos(s\omega) = (1/2) + \sin((m-1/2)\omega)/(2\sin(\omega/2))$ and $\sum_{s=0}^{m-1}\sin(s\omega) = \cos(\omega/2)/(2\sin(\omega/2)) - \cos((m-1/2)\omega)/(2\sin(\omega/2))$

References

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